Practice problem

Part 1: Complex number system

1. Find all the 10th roots of unity.
2. Find the roots of $(-27i)^{1/6}$, locate them graphically.
3. (a) Describe graphically the region represented by each of the following,

    $(i)\quad Re(1/z) > 1 \quad (ii)\quad Re(z^2) > 1 \quad (iii)\quad Im(z^2) = 4 \quad (iv)\quad 1 <\mid z + i \mid \leq 2$

4. Prove **De Moivre's theorem**.
    (a) And show that,

    $$\sin(0\theta) + \sin(\theta) + \cdots + \sin(n\theta) = \frac{\sin(\frac{n}{2}\theta)\sin(\frac{n+1}{2}\theta)}{\sin(\frac{1}{2}\theta)}$$

    (b) Find $\cos(5\theta)$ or $\sin(5\theta)$
5. Prove that $|z - 2i| + |z + 2i| = 6$ represents an ellipse.
6. Suppose we choose the principal branch of $\tanh^{-1} z$ to be that one for which $\tanh^{-1} 0 = 0$. Prove that,

    $$\tanh^{-1} z = \frac{1}{2} \ln\left(\frac{1+z}{1-z}\right)$$

7. Suppose we choose the principal branch of $\sin^{-1} z$ to be that one for which $\sin^{-1} 0 = 0$. Prove that,

    $$\sin^{-1} z = \frac{1}{i} \ln\left(iz + \sqrt{1 - z^2}\right)$$

8. Find the condition and show that,

    $$\cot^{-1} z = \frac{1}{2i} \ln\left(\frac{z+i}{z-i}\right)$$

9. Find the solution set of the equation,

    $$z^2 + 2z + (1 - i) = 0$$

10. Find x and y where,

    $$\left(\frac{3}{2} + \frac{\sqrt{3}i}{2}\right)^{102} = 3^{51}(5x + 2iy)$$

11. Solve for $z \in \mathbb{C}$,

$$z^5 + z^4 + z^3 + z^2 + z + 1 = 0$$

Part 2: Function

12. Find i^i and 1^i.

13. Find the image of the unit square in figure 1 under the mapping $f(z) = (1+i)z + i$,

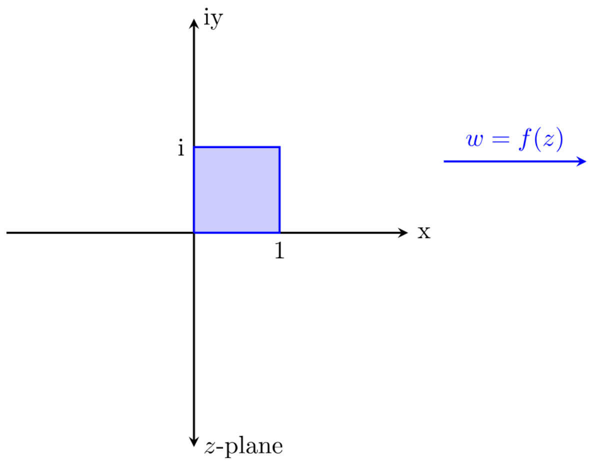

Figure 1: $f(z) = (1+i)z + i$

Part 3: Limit & Continuity

14. If $f(z) = \begin{cases} \frac{z^2-4}{z^2-3z+2}, z \neq 2 \\ kz^2 + 6, z = 2 \end{cases}$, find k such that the function $f(z)$ becomes continuous at $z = 2$.

15. Find

$$\lim_{z \to 0} \left(\frac{\sin z}{z} \right)^{\frac{1}{z^2}}$$

16. Show that the limit does not exist,

$$\lim_{z \to 0} \frac{\bar{z}}{z}$$

17. Show that the limit does not exist,

$$\lim_{z \to 0} \frac{xy}{x^2 + y^2}$$

Part 4: Differentiability & Harmonics

18. State and prove the **Cauchy-Riemann Theorem** or Let $A \subseteq \mathbb{C}$ be an open set, $z_0 \in A$, and let $f : A \to \mathbb{C}$ where $f = u + iv\,(u, v : \mathbb{R}^2 \to \mathbb{R})$. Then f is analytic at z_0 if and only there is a neighbourhood $\mathcal{N} \subseteq A$ of z_0 for which:

    1. $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}$, and $\frac{\partial v}{\partial y}$ all exist and are continuous on $\mathcal{N}$,
    2. The equations $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$, called the Cauchy-Riemann Equations, are satisfied on all of $\mathcal{N}$.

    Moreover we have that $f'(x + iy) = \frac{\partial u}{\partial x}(x, y) + i\frac{\partial v}{\partial x}(x, y) = \frac{\partial v}{\partial y}(x, y) - i\frac{\partial u}{\partial y}(x, y)$.

19. Using the definitions, find the derivative of the function at the indicated points,

$$f(z) = \frac{z - i}{z^2 + 1} \quad \text{at } z = i, -i$$

20. Show that, the derivative of $f(z) = |z|^2$ or $f(z) = z\bar{z}$ does not exist for $z \neq 0$.

21. Show that, if $f(z) = z - \bar{z}$ then $f'(z)$ does not exist.
    Answer: Use CR equations.

22. Show that $f(z) = \text{Re}(z)$ is nowhere differentiable.

23. Consider the function,

$$u(x, y) = e^{-x}(x \sin y - y \cos y)$$

    (a) Prove that, $u(x, y)$ is a harmonic function.
    (b) Find the harmonic conjugate function $v(x, y)$ such that $u(x, y) + iv(x, y)$ is analytic.

24. Given $f(z) = u + iv$ is analytic in a region $D \subseteq \mathbb{C}$. Prove that u and v are harmonic in D if they have continuous second partial derivatives in D.

25. Given that $f(z) = u + iv$ is analytic in a region R. Prove that u and v are harmonic if they have continuous second partial derivatives in R.

Best of Luck!