Complex variables & Laplace Transformation

Assignment 01

November 3, 2025

Total - 120 Marks

Due date: First Day after the Midterm

(You need to answer three questions from each section except Part-2. Answer both questions from Part-2)

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1. (a) Express each of the following complex numbers in polar form,

$$(i) \quad 2+2\sqrt{3}, \quad (ii) \quad -5+5i$$

   (b) Express each of the following complex numbers in Rectangular form,

$$(i) \quad 13\exp\left(\tan^{-1}\left(\frac{5}{12}\right)i\right), \quad (ii) \quad 4\left(\cos\frac{11}{6}\pi + i\sin\frac{11}{6}\pi\right) \quad \text{or} \quad r=4, \theta=-\frac{\pi}{3}$$

   (c) Find the principal argument of the complex numbers from (a).

   (d) Find all the 10th roots of unity.

   (e) Find the roots of $(-27i)^{1/6}$, locate them graphically.

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2. (a) Describe graphically the region represented by each of the following,

$$(i) \quad Re(1/z) > 1 \quad (ii) \quad Re(z^2) > 1 \quad (iii) \quad Im(z^2) = 4 \quad (iv) \quad |\, z-3 \,| - |\, z+3 \,| = 4$$

   (b) Solve the equation $z^2 + (2i-3)z + 5 - i = 0$

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3. Prove **De Moivre's theorem**.
   (a) And show that,

$$\sin(0\theta) + \sin(\theta) + \cdots + \sin(n\theta) = \frac{\sin(\frac{n}{2}\theta)\sin(\frac{n+1}{2}\theta)}{\sin(\frac{1}{2}\theta)}$$

   (b) Find $\cos(5\theta)$ or $\sin(5\theta)$

4. Prove that $|z - 2i| + |z + 2i| = 6$ represents an ellipse.
   Answer: Discussed in the classroom.

5. Suppose we choose the principal branch of $\tanh^{-1} z$ to be that one for which $\tanh^{-1} 0 = 0$. Prove that,

$$\tanh^{-1} z = \frac{1}{2} \ln\left(\frac{1+z}{1-z}\right)$$

6. Suppose we choose the principal branch of $\sin^{-1} z$ to be that one for which $\sin^{-1} 0 = 0$. Prove that,

$$\sin^{-1} z = \frac{1}{i} \ln\left(iz + \sqrt{1-z^2}\right)$$

7. Find the condition and show that,

$$\cot^{-1} z = \frac{1}{2i} \ln\left(\frac{z+i}{z-i}\right)$$

8. Find the solution set of the equation,

$$z^2 + 2z + (1 - i) = 0$$

9. Find x and y where,

$$\left(\frac{3}{2} + \frac{\sqrt{3}i}{2}\right)^{102} = 3^{51}(5x + 2iy)$$

10. Solve for $z \in \mathbb{C}$,

$$z^5 + z^4 + z^3 + z^2 + z + 1 = 0$$

Part 2: Function

11. Find i^i and 1^i.

12. (a) Find the image of the given line under the given map,

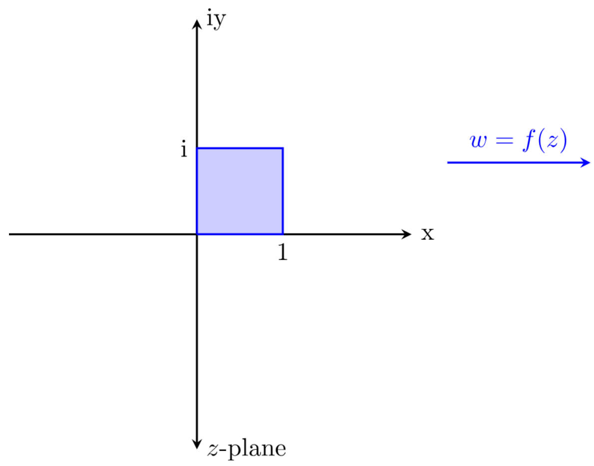

Figure 1: $f(z) = (1+i)z + i$

1. $Im(z) = 1; \quad f(z) = \cos(z)$
2. $Re(z) = \frac{\pi}{6}; \quad f(z) = \sin(z)$

13. Find the image of the unit square in figure 1 under the mapping $f(z) = (1+i)z + i$,

Part 3: Limit & Continuity

14. If $f(z) = \begin{cases} \frac{z^2-4}{z^2-3z+2}, z \neq 2 \\ kz^2+6, z = 2 \end{cases}$, find k such that the function $f(z)$ becomes continuous at $z = 2$.

15. Find

$$\lim_{z \to 0} \left(\frac{\sin z}{z} \right)^{\frac{1}{z^2}}$$

16. Show that the limit does not exist,

$$\lim_{z \to 0} \frac{\bar{z}}{z}$$

17. Show that the limit does not exist,

$$\lim_{z \to 0} \frac{xy}{x^2+y^2}$$

Part 4: Differentiability & Harmonics

18. State the **Cauchy-Riemann Theorem** or Let $A \subseteq \mathbb{C}$ be an open set, $z_0 \in A$, and let $f : A \to \mathbb{C}$ where $f = u + iv\,(u, v : \mathbb{R}^2 \to \mathbb{R})$. Then f is analytic at z_0 if and only there is a neighbourhood $\mathcal{N} \subseteq A$ of z_0 for which:

    1. $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}$, and $\frac{\partial v}{\partial y}$ all exist and are continuous on $\mathcal{N}$,
    2. The equations $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$, called the Cauchy-Riemann Equations, are satisfied on all of $\mathcal{N}$.

    Moreover we have that $f'(x + iy) = \frac{\partial u}{\partial x}(x, y) + i\frac{\partial v}{\partial x}(x, y) = \frac{\partial v}{\partial y}(x, y) - i\frac{\partial u}{\partial y}(x, y)$.

19. Using the definitions, find the derivative of the function at the indicated points,

$$f(z) = \frac{z - i}{z^2 + 1} \quad \text{at } z = i, -i$$

20. Show that, the derivative of $f(z) = |z|^2$ or $f(z) = z\bar{z}$ does not exist for $z \neq 0$.

21. Show that, if $f(z) = z - \bar{z}$ then $f'(z)$ does not exist.

22. Show that $f(z) = \text{Re}(z)$ is nowhere differentiable.

23. Consider the function,

$$u(x, y) = e^{-x}(x \sin y - y \cos y)$$

    (a) Prove that, $u(x, y)$ is a harmonic function.

    (b) Find the harmonic conjugate function $v(x, y)$ such that $u(x, y) + iv(x, y)$ is analytic.

24. Given $f(z) = u + iv$ is analytic in a region $D \subseteq \mathbb{C}$. Prove that u and v are harmonic in D if they have continuous second partial derivatives in D.

Best of Luck!