Contents

1 The Complex Number System

Complex number's day-to-day application is not as direct as that of real numbers, their imaginary component makes complex numbers important as they make it possible to work very precisely in specific areas of science and physics. This is the case with measuring electromagnetic fields, which consist of electrical and magnetic components and require pairs of real numbers to describe them. These pairs can be seen as a complex number, hence their importance.

§1.1 The Real Number System

The number system as we know it today is a result of gradual development as indicated in the following list.

1. Natural Numbers: $1, 2, 3, 4, \ldots\ldots.$
2. Negative integers and zero: $0, -1, -2, -3, -4, \ldots\ldots$
3. Rational Numbers: $\left\{x \mid x \text{ is in the form } \frac{a}{b}, b \neq 0\right\}$
4. Irrational Numbers: $\left\{x \mid x \text{ is not in the form } \frac{a}{b}, b \neq 0\right\}$
5. Real Number: $\{\text{Rational Number Set }\} \cup \{\text{Irrational Number }\}$

Real numbers can be represented by points on a line called the real axis, as indicated in Figure 1.1. The point corresponding to zero is called the origin.

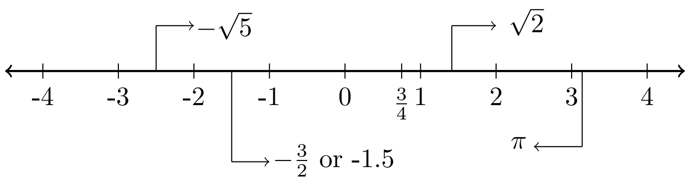


Figure 1.1

§1.2 The Complex Number System

There is no real number x that satisfies the polynomial equation $x^2 + 1 = 0$. To permit solutions of this and similar equations, the set of complex numbers is introduced.

Definition 1.1. A complex number takes the form $z = a + ib$ where a and b are real, and i is an imaginary number that satisfies $i^2 = -1$. We call a and b the real part and the imaginary part of z, respectively, and we write

$$a = \text{Re}(z) \quad \text{and} \quad b = \text{Im}(z).$$

The real numbers are precisely those complex numbers with zero imaginary parts.

Definition 1.2. A complex number with zero real part is said to be purely imaginary.

Theorem 1.1

Two complex numbers $a+bi$ and $c+di$ in Cartesian coordinate system are equal if and only if $a=c$ and $b=d$.

We can consider real numbers as a subset of the set of complex numbers with $b=0$. Accordingly the complex numbers $0+0i$ and $-3+0i$ represent the real numbers 0 and -3 , respectively. If $a=0$, the complex number $0+bi$ or bi is called a pure imaginary number.

Definition 1.3 (Complex Conjugate). The complex conjugate, or briefly conjugate, of a complex number $a+bi$ is $a-bi$. The complex conjugate of a complex number z is often indicated by $\bar{z}$ or z^*.

$$\begin{aligned} z=a+ib \quad &\longrightarrow \quad \text{(Complex Number)} \\ \bar{z}=a-ib \quad &\longrightarrow \quad \text{(Complex conjugate of } z \text{)} \end{aligned}$$

§1.3 Fundamental Operations of Complex Numbers

In performing operations with complex numbers, we can proceed as in the algebra of real numbers, replacing i^2 by -1 or i^3 by $-i$ or i^4 by 1 when it occurs and so on.

Corollary 1.2

Addition

$$(a+bi)+(c+di)=a+bi+c+di=(a+c)+(b+d)i$$

Corollary 1.3

Subtraction

$$(a+bi)-(c+di)=a+bi-c-di=(a-c)+(b-d)i$$

Corollary 1.4

Multiplication

$$(a+bi)(c+di)=ac+adi+bci+bdi^2=(ac-bd)+(ad+bc)i$$

Corollary 1.5

Division

If $c \neq 0$ and $d \neq 0$, then

$$\begin{aligned} \frac{a+bi}{c+di} &= \frac{a+bi}{c+di} \cdot \frac{c-di}{c-di} = \frac{ac-adi+bci-bdi^2}{c^2-d^2i^2} \\ &= \frac{ac+bd+(bc-ad)i}{c^2+d^2} = \frac{ac+bd}{c^2+d^2} + \frac{bc-ad}{c^2+d^2}i \end{aligned}$$

Definition 1.4 (Absolute Value). The absolute value or modulus of a complex number $a+bi$ is defined as $|a+bi|=\sqrt{a^2+b^2}$.

Example 1.1. $|-4+2i|=\sqrt{(-4)^2+(2)^2}=\sqrt{20}=2\sqrt{5}$

Corollary 1.6

If $z_1, z_2, z_3, \ldots, z_m$ are complex numbers, the following properties hold.
(1) $|z_1 z_2| = |z_1|\,|z_2|$ or $|z_1 z_2 \cdots z_m| = |z_1|\,|z_2| \cdots |z_m|$
(2) $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$ if $z_2 \neq 0$
(3) $|z_1 + z_2| \leq |z_1| + |z_2|$ or $|z_1 + z_2 + \cdots + z_m| \leq |z_1| + |z_2| + \cdots + |z_m|$
(4) $|z_1 \pm z_2| \geq |z_1| - |z_2|$

§1.4 Graphical Representation of Complex Numbers

Okay, to visualize complex numbers in the complex plane we have several ways:

1. Rectangular form
2. Polar form
3. Exponential form

Throughout our presentation, the set of all complex numbers is denoted by $\mathbb{C}$. The complex numbers can be visualized as the usual Euclidean plane by the following simple identification: the complex number $z = x + iy \in \mathbb{C}$ is identified with the point $(x, y) \in \mathbb{R}^2$. For example, 0 corresponds to the origin and i corresponds to $(0, 1)$. Naturally, the x and y axis of $\mathbb{R}^2$ are called the real axis and imaginary axis, because they correspond to the real and purely imaginary numbers, respectively.

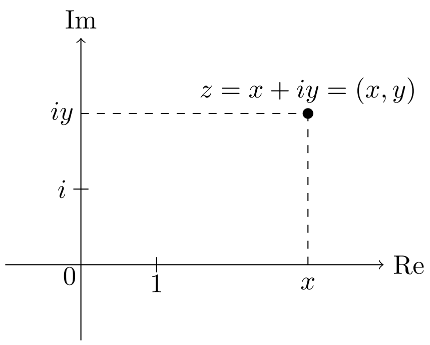


Figure 1.2: The Complex Plane

Definition 1.5. The distance between two complex numbers, $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ is,

$$|z_1 - z_2| = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

Example 1.2. The distance between $z_1 = 4 - 5i$ and $z_2 = -i$ is

$$\begin{aligned} |z_1 - z_2| &= \sqrt{(4-0)^2 + (-5+1)^2} \\ &= \sqrt{16+16} = \sqrt{32} = 4\sqrt{2} \end{aligned}$$

§1.5 Polar Form of Complex Number

Any non-zero complex number z can be written in polar form

$$z = re^{i\theta},$$

where $r > 0$; also $\theta \in \mathbb{R}$ is called the argument of z (defined uniquely up to a multiple of 2π) and is often denoted by $\arg z$, and

$$e^{i\theta} = \cos\theta + i\sin\theta.$$

Since $|e^{i\theta}| = 1$ we observe that $r = |z|$, and θ is simply the angle (with positive counterclockwise orientation) between the positive real axis and the half-line starting at the origin and passing through z.

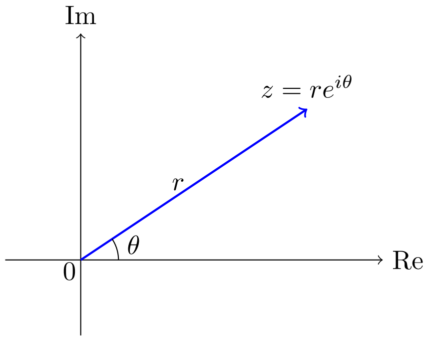


Figure 1.3: The polar form of a complex number

Theorem 1.7

Two complex numbers $z_1 = r_1 e^{i\theta_1}$ and $z_2 = r_2 e^{i\theta_2}$ in the polar coordinate will be equal if

$$r_1 = r_2 \qquad \theta_1 = \theta_2 + 2k\pi \qquad \text{where } k = 0, \pm1, \pm2, \pm3, \ldots\ldots\ldots$$

Definition 1.6. In order to make the argument of z a well-defined number, it is sometimes restricted to the interval $(-\pi, \pi]$. This special choice is called the principal value or the main branch of the argument and is written as $Arg(z)$.

§1.6 Finding Principal Argument

Consider a complex number $z = x + iy$ or (x, y) on the complex plane. Define, $\alpha = \tan^{-1}\left(\left|\frac{y}{x}\right|\right)$

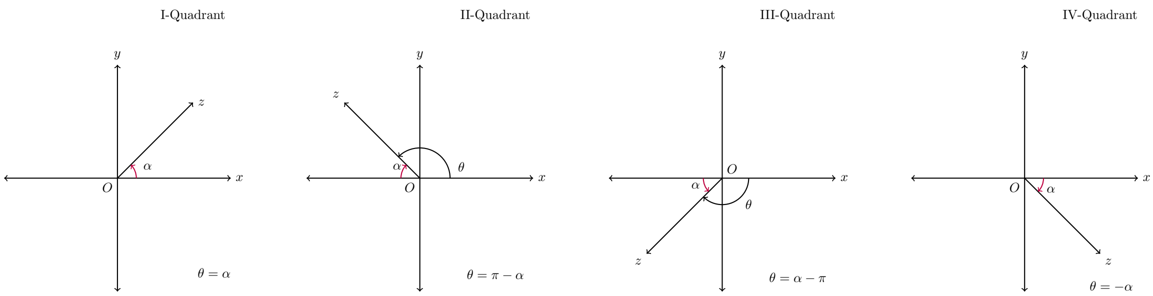


Figure 1.4: Principle Argument of Complex Number

$(x, y) \rightarrow$ coordinate of the complex number in 2D complex plane.
$(\pi, \theta) \rightarrow$ polar coordinate of the complex number in the complex plane.
Where,

$$r = \sqrt{x^2 + y^2} = |x + iy| = \text{ modulus or absolute value of } z = x + iy$$

and $\theta = \text{Arg}(z)$
Then a complex number of the form $z = x + iy$ can be represented using polar coordinate as,

$$\begin{aligned} z = x + iy &= r\cos\theta + ir\sin\theta \\ &= r(\cos\theta + i\sin\theta) \end{aligned}$$

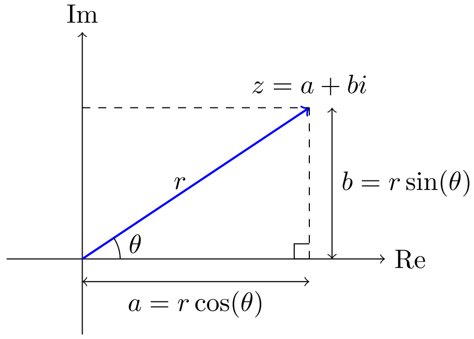


Figure 1.5

Example 1.3. Find the principal argument of the following complex numbers.
(a) $1 - i$ (b) $2 + 2\sqrt{3}i$ (c) 4i

Solution:

(a) Principal argument, $\text{Arg}(z) = -\tan^{-1}\left(\left|\frac{y}{x}\right|\right)$

$$= -\tan^{-1}(1)$$
$$= -\frac{\pi}{4}$$

(b) Principal argument, $\text{Arg}(z) = +\tan^{-1}\left(\left|\frac{y}{x}\right|\right)$

$$= \tan^{-1}\left(\frac{2\sqrt{3}}{2}\right)$$
$$= \tan^{-1}(\sqrt{3}) = \frac{\pi}{3}$$

(c) Principal argument, $\text{Arg}(z) = \tan^{-1}\left(\frac{4}{0}\right) = \frac{\pi}{2}$

Example 1.4. Consider two complex number $z_1 = -i$, $z_2 = -i + 4$
(i) Plot z_1 and z_2
(ii) Find the modulus and principal argument of z_1 and z_2

$$\left[\textbf{Answer: } r_1 = 1, \theta_1 = -\pi/2; \quad r_2 = \sqrt{17}, \theta_2 = \pi - \tan^{-1}\left(\frac{1}{4}\right)\right]$$

(iii) Show that, $|z_1| = |\bar{z}_1|$, where $\bar{z}_1$ is the complex conjugate of z_1.

(iv) Find, $z_1 z_2$ and $\frac{z_1}{z_2}$

Example 1.5. Convert $z = 4e^{-i\frac{\pi}{3}}$ in Cartesian coordinate system.
Solution: Here, $\pi = 4$, $\theta = -\frac{\pi}{3}$

$$x = r\cos\theta = 4\cos\left(-\frac{\pi}{3}\right) = 4\frac{1}{2} = 2$$
$$y = r\sin\theta = 4\sin\left(-\frac{\pi}{3}\right) = -4\frac{\sqrt{3}}{2} = -2\sqrt{3}$$

So, in Cartesian form, $z = 2 - 2\sqrt{3}i$.

Example 1.6. Convert $z = \sqrt{3} + 2\sqrt{3}i$ in the polar form.

§1.7 De Moiver's Theorem

Let $z_1 = x_1 + iy_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = x_2 + iy_2 = r_2(\cos\theta_2 + i\sin\theta_2)$, then we can show that

$$z_1 z_2 = r_1 r_2 \{\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)\} \tag{1.1}$$
$$\frac{z_1}{z_2} = \frac{r_1}{r_2}\{\cos(\theta_1-\theta_2) + i\sin(\theta_1-\theta_2)\} \tag{1.2}$$

A generalization of (1.1) leads to

$$z_1 z_2 \cdots z_n = r_1 r_2 \cdots r_n \{\cos(\theta_1+\theta_2+\cdots+\theta_n) + i\sin(\theta_1+\theta_2+\cdots+\theta_n)\}$$

and if $z_1 = z_2 = \cdots = z_n = z$ this becomes

$$z^n = \{r(\cos\theta + i\sin\theta)\}^n = r^n(\cos n\theta + i\sin n\theta)$$

which is often called De Moivre's theorem.

§1.8 nth-Root of Complex Numbers

Any nonzero complex number has exactly $n \in N$ distinct n-th roots. The roots lie on a circle of radius $|z|$ centered at the origin and spaced out equally by angles of $\dfrac{2\pi}{n}$.

> **Definition 1.7.** A number w is called n-th root of the complex number z if $\omega^n = z$ and we can write $\omega = z^{\frac{1}{n}}$.

Example 1.7. Find the n-th root of the complex number of the form $z = \pi e^{i\theta}$
Solution: Let $z_0 = r_0 e^{i\theta_0}$ be the n-th root of z. So,

$$\begin{aligned}
&z_0^n = z\\
\Rightarrow &z_0^n = re^{i\theta}\\
\Rightarrow &\left(r_0 e^{i\theta_0}\right)^n = re^{i\theta}\\
\Rightarrow &r_0^n e^{in\theta_0} = re^{i\theta}
\end{aligned}$$

which implies that, $r_0^n = r \Rightarrow r_0 = r^{\frac{1}{n}}$
and $n\theta_0 = \theta + 2k\pi \quad \{$ where $k = 0, \pm1, \pm2, \ldots\}$
Now,

$$\begin{aligned}
&\text{for } k=0, \quad n\theta_0 = \theta \Rightarrow \theta_0 = \frac{\theta}{n}\\
&\text{for } k=1, \quad n\theta_0 = \theta + 2\pi \Rightarrow \theta_0 = \frac{\theta}{n} + \frac{2\pi}{n}\\
&\text{for } k=2, \quad n\theta_0 = \theta + 4\pi \Rightarrow \theta_0 = \frac{\theta}{n} + \frac{4\pi}{n}\\
&\text{for } k=n-1, \quad n\theta_0 = \theta + 2\pi(n-1) \Rightarrow \theta_0 = \frac{\theta}{n} + \frac{2\pi(n-1)}{n}\\
&\text{for } k=n, \quad n\theta_0 = \theta + 2n\pi \Rightarrow \theta_0 = \frac{\theta}{n} + 2\pi && \text{(already found)}\\
&\text{for } k=n+1, \quad n\theta_0 = \theta + 2\pi(n+1) \Rightarrow \theta_0 = \frac{\theta}{n} + \frac{2\pi}{n} + 2\pi && \text{(already found)}\\
&\quad\vdots\\
&\text{for } k=-1, \quad n\theta_0 = \theta - 2\pi \Rightarrow \theta_0 = \frac{\theta}{n} + \frac{2\pi(n-1)}{n} && \text{(already found)}\\
&\quad\vdots\\
&\text{and so on}
\end{aligned}$$

As we are getting same root repeatedly, distinct n-roots are,

$$\left.\begin{array}{l} k=0 \longrightarrow z_1 = r^{1/n} exp\left(i\frac{\theta}{n}\right) \\ k=1 \longrightarrow z_2 = r^{1/n} exp\left(i\frac{2\pi+\theta}{n}\right) \\ k=2 \longrightarrow z_3 = r^{1/n} exp\left(i\frac{4\pi+\theta}{n}\right) \\ \vdots \\ k=(n-2) \longrightarrow z_{n-1} = r^{1/n} exp\left(i\frac{2\pi(n-2)+\theta}{n}\right) \\ k=(n-1) \longrightarrow z_n = r^{1/n} exp\left(i\frac{2\pi(n-1)+\theta}{n}\right) \end{array}\right\}$$

Example 1.8. (i) Find all values of z such that $z^5 = -32$. (ii) Locate these values in the complex plane.
Solution:
(i) The polar form of the given complex number, $z = -32 = 32e^{i\pi}$
So,

$$\begin{aligned} & z_0^5 = -32 \\ \Rightarrow & (r_0 e^{i\theta_0})^5 = 32e^{i\pi} \\ \Rightarrow & r_0^5 e^{i5\theta_0} = 32e^{i\pi} \end{aligned}$$

Thus, $r_0 = 2$
and $5\theta_0 = \pi + 2\pi k$, where $k = 0, 1, 2, 3, 4$
Thus we will get the 5th root as follows,

$$r_0 e^{\frac{\pi+2\pi k}{5}} \qquad \text{where } k = 0, 1, 2, 3, 4$$

More specifically,

$$2exp\left(i\frac{\pi}{5}\right), \quad 2exp\left(i\frac{3\pi}{5}\right), \quad 2exp\left(i\frac{5\pi}{5}\right), \quad 2exp\left(i\frac{7\pi}{5}\right), \quad \text{and} \quad 2exp\left(i\frac{9\pi}{5}\right)$$

(ii) Thus, the graphical representation of the roots are,

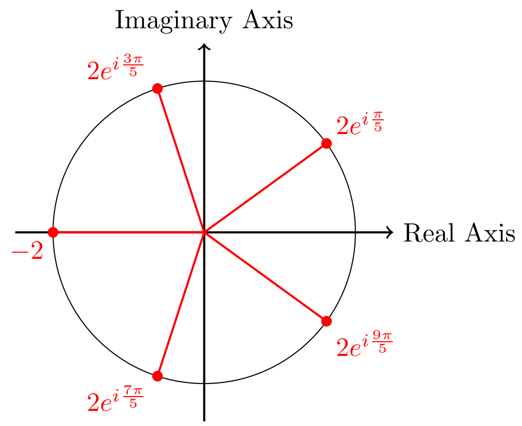


Figure 1.6: Graphical representation of 5 roots

Example 1.9. Find the 4-th root of $-2\sqrt{3} - 2i$ and locate them graphically.
Solution:
The polar form of the given complex number, $z = -2\sqrt{3} - 2i = 4e^{-i2\pi/3}$
So,

$$\begin{aligned} & z_0^4 = -2\sqrt{3} - 2i \\ \Rightarrow & (r_0 e^{i\theta_0})^4 = 4e^{-i2\pi/3} \\ \Rightarrow & r_0^4 e^{i4\theta_0} = 4e^{-i2\pi/3} \end{aligned}$$

Thus, $r_0 = \sqrt{2}$
and $4\theta_0 = -2\pi/3 + 2\pi k$, where $k = 0, 1, 2, 3$
Thus we will get the 4th root as follows,

$$r_0 e^{\frac{-2\pi/3+2\pi k}{4}} \qquad \text{where } k = 0, 1, 2, 3$$

More specifically,

$$\sqrt{2}exp\left(-i\frac{\pi}{6}\right), \quad \sqrt{2}exp\left(i\frac{\pi}{3}\right), \quad \sqrt{2}exp\left(i\frac{5\pi}{6}\right), \quad \text{and} \quad \sqrt{2}exp\left(i\frac{4\pi}{3}\right)$$

Graphically,

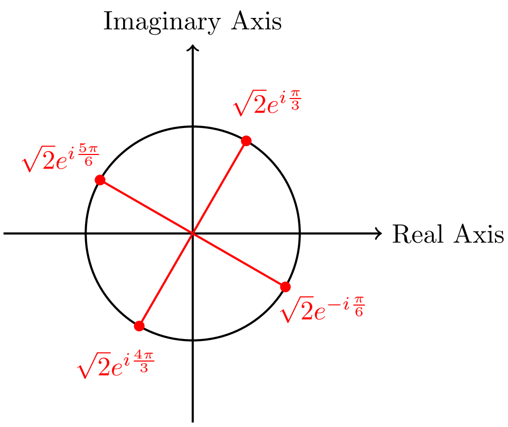


Figure 1.7: Graphical representation of 4 roots

Example 1.10. Find the 6-th root of $-1 + \sqrt{3}i$ and locate them graphically.
Solution:
The polar form of the given complex number, $z = -1 + \sqrt{3}i = 2e^{i2\pi/3}$
So,

$$z_0^6 = -1 + \sqrt{3}i$$
$$\Rightarrow (r_0 e^{i\theta_0})^6 = 2e^{i2\pi/3}$$
$$\Rightarrow r_0^6 e^{i6\theta_0} = 2e^{i2\pi/3}$$

Thus, $r_0 = 2^{(1/6)}$
and $6\theta_0 = 2\pi/3 + 2\pi k$, where $k = 0, 1, 2, 3, 4, 5$
Thus we will get the 6th root as follows,

$$r_0 e^{\frac{2\pi/3+2\pi k}{6}} \qquad \text{where } k = 0, 1, 2, 3, 4, 5$$

More specifically,

$$2^{(1/6)}exp\left(i\frac{\pi}{9}\right), \quad 2^{(1/6)}exp\left(i\frac{4\pi}{9}\right), \quad 2^{(1/6)}exp\left(i\frac{7\pi}{9}\right),$$
$$2^{(1/6)}exp\left(i\frac{10\pi}{9}\right), \quad 2^{(1/6)}exp\left(i\frac{13\pi}{9}\right), \quad \text{and} \quad 2^{(1/6)}exp\left(i\frac{16\pi}{9}\right)$$

Graphically,

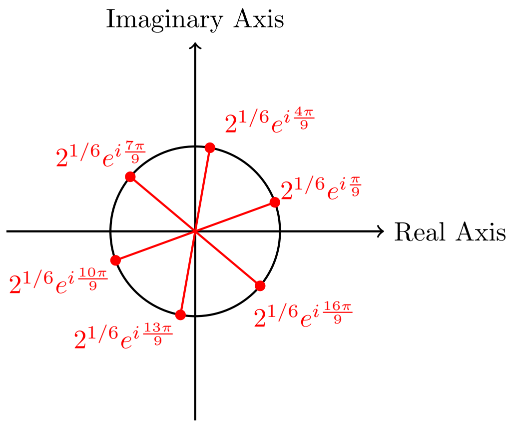


Figure 1.8: Graphical representation of 4 roots

§1.9 Graphical Representation of Complex Regions

In this explainer, we will learn how to identify regions in the complex plane.

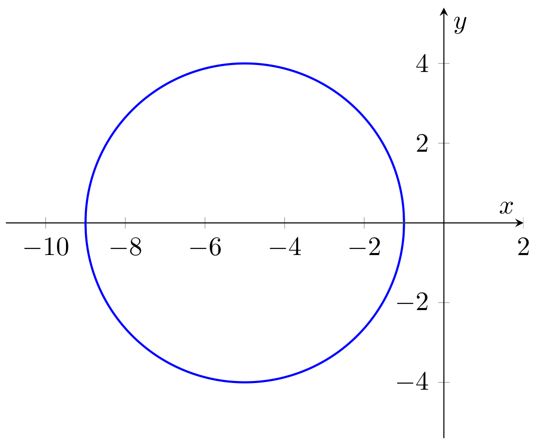
Example 1.11. Describe each of the region graphically,

$$(a) \quad \left|\frac{z-3}{z+3}\right| = 2 \quad (b) \quad \left|\frac{z-3}{z+3}\right| < 2$$

Solution:
(a)

$$\left|\frac{z-3}{z+3}\right| = 2$$
$$\Rightarrow \mid z-3 \mid = 2 \mid z+3 \mid$$
$$\Rightarrow (x+5)^2 + y^2 = 16 \quad \text{[After substituting } z = x+iy\text{]}$$

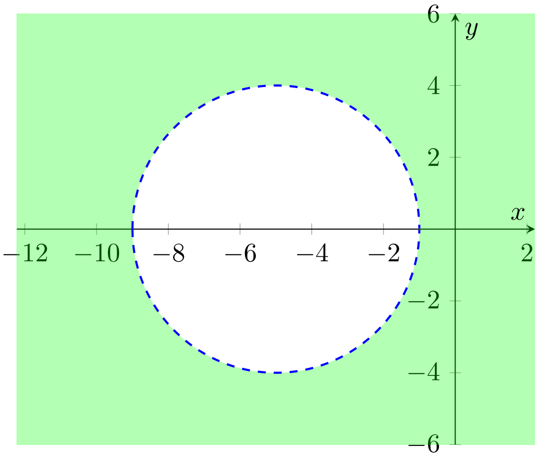
Graphically,



(b)

$$\left|\frac{z-3}{z+3}\right| < 2$$
$$\Rightarrow (x+5)^2 + y^2 > 16$$

Graphically,



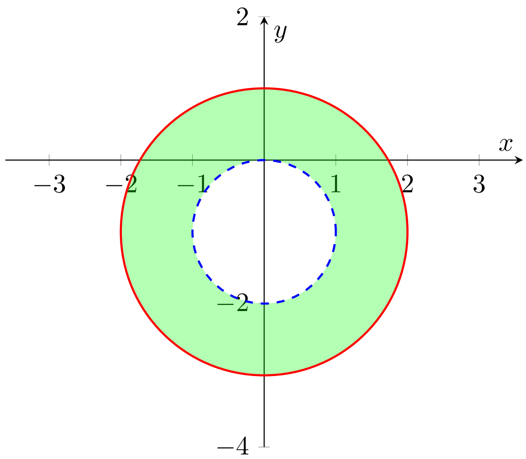
Example 1.12. Describe each of the region graphically,

$$(a) \quad 1 < \mid z+i \mid \leq 2 \quad (b) \quad Re(z^2) > 1 \quad (c) \quad Im(z^2) < 4$$

Solution:
(a)

$$\begin{aligned} & 1 < \mid z+i \mid \leq 2 \\ \Rightarrow \quad & 1 < \mid x+iy+i \mid \leq 2 \\ \Rightarrow & 1 < \sqrt{x^2+(1+y)^2} \leq 2 \\ \Rightarrow & 1^2 < x^2+(1+y)^2 \leq 2^2 \end{aligned}$$

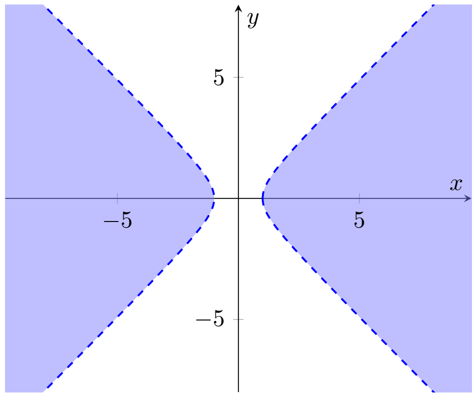
Graphically,



(b)

$$\begin{aligned} & Re(z^2) > 1 \\ \Rightarrow & Re(x^2-y^2+i2xy) > 1 \\ \Rightarrow & x^2-y^2 > 1 \end{aligned}$$

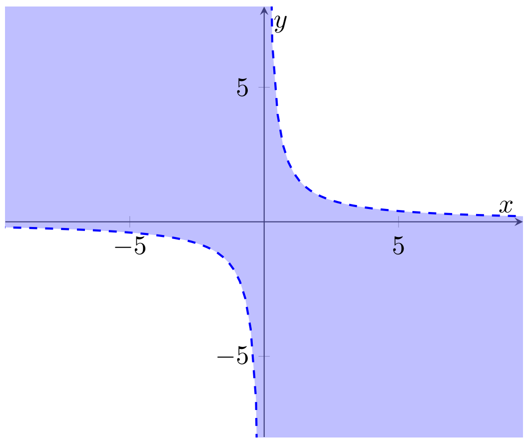
Graphically,



(c)

$$\begin{aligned} &Im(z^2) < 4 \\ \Rightarrow &Im(x^2 - y^2 + i2xy) < 4 \\ \Rightarrow &xy < 2 \end{aligned}$$

Graphically,



§1.10 Exercise

Exercise 1.1. Express each of the following complex numbers in polar form,

$$(a) \quad 2 + 2\sqrt{3}, \quad (b) \quad -5 + 5i, \quad \text{and} \quad (iii) \quad -\sqrt{6} - \sqrt{2}i$$

Exercise 1.2. Find the square root of $-15 - 8i$.

Exercise 1.3. Solve the equation $z^2 + (2i - 3)z + 5 - i = 0$

Exercise 1.4. Find all the 10-th root of unity.

Exercise 1.5. Find the indicated roots and locate them graphically,

$$(a) \quad (64)^{1/6} \quad (b) \quad (i)^{2/3} \quad (c) \quad (-1+i)^{1/3} \quad (d) \quad (-27i)^{1/6} \quad (e) \quad (-11-2i)^{1/3}$$

Exercise 1.6. Describe each of the region graphically,

$$(i) \quad Re(z) > 1 \quad (ii) \quad |\, 2z + 3 \,| > 4 \quad (iii) \quad 1 < |\, z - 2i \,|\, 2 \quad (iv) \quad |\, z + 1 - i \,| \le |\, z - 1 + i \,|$$

$$(v) \quad Re(1/z) > 1 \quad (vi) \quad |\, z - 4 \,| \ge |\, z \,| \quad (vii) \quad Re(1/z) \le \frac{1}{2} \quad (viii) \quad |\, z - 2 \,| \le |\, z + 2 \,|$$

Exercise 1.7. Using the properties of Conjugate and modulus to show that,

$$| 2z + 3\bar{z} | \leq 4 \mid Re(z) \mid + \mid z \mid$$

Exercise 1.8. Prove that, $| z + 2i | + | z - 2i | = 6$ represents an ellipse.

Exercise 1.9. Describe graphically the region represented by each of the following,

$$(i) \quad 1 < | z + i | \leq 2 \quad (ii) \quad Re(z^2) > 1 \quad (iii) \quad Im(z^2) = 4 \quad (iv) \quad | z - 3 | - | z + 3 | = 4$$

§1.11 Complementary of chapter one

§1.11.i $\mathbb{R}^2 \cong \mathbb{C}$

Okay, we already see why real numbers can't help us to get the solution of $x^2 + 1 = 0$. Because while solving the equation we get something very strange! $x = \sqrt{-1}$. Let's suppose, we want to integrate this somewhere in our well familiar real number line by denoting it $i = \sqrt{-1}$. Then subsequent problems arise for $x^2 + 2, x^2 + 3, \cdots$, ummm. Aren't we just copying everything with the magical symbol i. That's where we found $i\mathbb{R}$. Now, if we want to combine everything, we just land in $\mathbb{R} + i\mathbb{R}$. That's what we call complex plane $\mathbb{C}$. If your linear algebra sense turns in then you will say, that $\mathbb{C}$ is nothing but

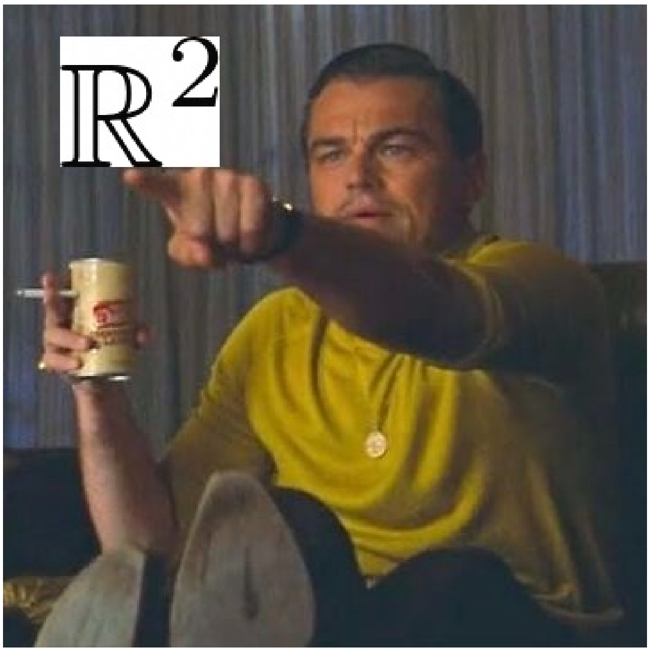


Figure 1.9: Imposter

Now, the fun part arises. We are saying $\mathbb{C} \cong \mathbb{R}^2$. but they are very different. Like, one is a vector space, another is a field. Not only $\mathbb{C}$ inherit all nicer properties of $\mathbb{R}^2$, but also has some stronger structure. $\mathbb{C}$ has no order. Let's assume it has.

Let $<$ be any arbitrary total ordering on $\mathbb{C}$. Then $i \neq 0$ gives, either $i < 0$ or $i > 0$. But we will show none of them holds true.
If $i > 0$ then from the condition (2) we get, $i \cdot i > i \cdot 0 \implies -1 > 0$. Now some may think that we have arrived at a contradiction but unfortunately no. Since $<$ is an arbitrary ordering this may happen. But apply condition (2) again and we get, $(-1) \cdot i > 0 \cdot i \implies -i > 0$. Now using condition (1), $i > 0$ and $-i > 0 \implies i + (-i) > 0 + 0 \implies 0 > 0$. Which is a contradiction.
Similarly if we put $i < 0$ then from condition (2) we get, $i \cdot i > 0 \cdot i \implies -1 > 0$. Then again apply condition (2) on $i < 0$ to get, $i \cdot (-1) < 0 \cdot (-1) \implies -i < 0$. Again using condition (1), $i > 0$ and $-i > 0 \implies i + (-i) > 0 + 0 \implies 0 > 0$. Which is a contradiction.
Hence i and 0 are not comparable, so there is no total order on $\mathbb{C}$ which makes it an ordered field.

We are mathematicians, right? Let's force $\mathbb{R}^2$ to be a field. Okay, then we must have a multiplication notion. Let's define one:

$$\begin{pmatrix} a \\ b \end{pmatrix} \odot \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} ac - bd \\ ad + bc \end{pmatrix}$$

Aha! So far so good and it's a field now. But, now we will show that,

Theorem 1.8

There must exist some $z \in \mathbb{R}^2$ such that $z^2 = -1$.

Proof. We shall construct such z. As $\mathbb{R}^2$ is a 2-dimensional vector space, it is spanned by a basis set with 2 elements. Let the basis set be $\{\mathbf{1}, \mathbf{e}\}$. Here $\mathbf{1}$ basically means the pair $(1, 0)$. Take $z \in \mathbb{R}^2$ such that it's not on the x-axis. It's an element of the vector space so that it can be written as a linear combination of the bases. That is,

$$z = x \cdot \mathbf{1} + y \cdot \mathbf{e} \text{ where } x, y \in \mathbb{R} \text{ and } y \neq 0$$

Then we can calculate z^2, using the identity $(p+q)^2 = p^2 + q^2 + 2pq$.

$$z^2 = (x \cdot \mathbf{1} + y \cdot \mathbf{e})^2 = x^2 \cdot \mathbf{1} + y^2 \cdot \mathbf{e}^2 + 2xy \cdot \mathbf{e}$$

$\mathbf{e}^2 \in \mathbb{R}^2$, so it can be written as a linear combination of the bases. Let $\mathbf{e}^2 = a \cdot \mathbf{1} + b \cdot \mathbf{e}$. Plugging this, we get

$$\begin{aligned} z^2 &= x^2 \cdot \mathbf{1} + y^2 \cdot \mathbf{e}^2 + 2xy \cdot \mathbf{e} \\ &= x^2 \cdot \mathbf{1} + ay^2 \cdot \mathbf{1} + by^2 \cdot \mathbf{e} + 2xy \cdot \mathbf{e} \\ &= \left(x^2 + ay^2\right) \cdot \mathbf{1} + \left(by^2 + 2xy\right) \cdot \mathbf{e} \end{aligned}$$

Now, we choose x such that $by^2 + 2xy$ becomes 0. In other words, we choose $x = \frac{-by}{2}$ (we shall fix y later). So we have,

$$z^2 = \left(\left(\frac{-by}{2}\right)^2 + ay^2\right) \cdot \mathbf{1} + 0 \cdot \mathrm{e} = \left(a + \frac{b^2}{4}\right) y^2 \cdot \mathbf{1}$$

Claim: $a + \frac{b^2}{4} < 0$.

Proof. Assume for the sake of contradiction that $a + \dfrac{b^2}{4} \geq 0$. Then we have a notion of the square root of non-negative real numbers. So let $c = \sqrt{a + \dfrac{b^2}{4}}$. Now we have,

$$\begin{aligned} z^2 = c^2 y^2 \cdot \mathbf{1} &\implies z^2 - c^2 y^2 \cdot \mathbf{1}^2 = \mathbf{0} \\ &\implies (z - cy \cdot \mathbf{1})(z + cy \cdot \mathbf{1}) = \mathbf{0} \\ &\implies z - cy \cdot \mathbf{1} = \mathbf{0} \text{ or } z + cy \cdot \mathbf{1} = \mathbf{0} \\ &\implies z = cy \cdot \mathbf{1} \text{ or } z = -cy \cdot \mathbf{1} \end{aligned}$$

They both contradict the assumption that z does not lie on the x-axis. ■

So we have $-\left(a + \dfrac{b^2}{4}\right) > 0$. Let $c = \sqrt{-\left(a + \frac{b^2}{4}\right)}$. Taking $y = \dfrac{1}{c}$, we get

$$z^2 = -c^2 \frac{1}{c^2} \cdot \mathbf{1} = -\mathbf{1}$$

as desired. ■

So if we really wish to give $\mathbb{R}^2$ a field structure, then we must have some i in our space such that $i^2 = -1$.

§1.11.ii Matrix representation of $\mathbb{C}$

Figure 1.10: i

2 General Functions of a Complex Variable

If you want to study some space, study how function behaves there. While considering $\mathbb{C}$ as a vector space $\mathbb{R}^2$ with some more structure, we can see how a general function may look in a complex plane. Let's say, $f : \mathbb{C} \to \mathbb{C}$ where $z \mapsto w = f(z)$. Then if we want to plot the input-output pair, $(z, f(z))$ isn't we need 4 dimensions, $(2\,\mathrm{dim}, 2\,\mathrm{dim})$? Don't worry, there are 5 different ways to solve this problem and visualize the complex functions:

- Domain Coloring
- $3D$ Plots
- Vector Fields
- $z - w$ Planes
- Riemann Sphere

In the Domain Coloring method, we associate the output with a color representing the complex number $f(z)$. Where,

$$\text{Hue} \leftrightarrow \text{Argument}$$
$$\text{Lightness} \leftrightarrow \text{Modulus}$$

In the $3D$ Plots method, we sacrifice one real variable from $(z, f(z)) = (x, y, u, v)$ and associate the missing variable with a color representing the missing complex number v.

$$\text{Missing } v \leftrightarrow \text{color}$$

For more details have a look at https://www.youtube.com/watch?v=NtoIXhUgqSk.

Definition 2.1. A multi-valued function f on $E \subset \mathbb{C}$ assigns a set of complex values to each $z \in E$, i.e. $f(z)$ is a set of complex numbers.

Definition 2.2. A branch of a multi-valued function f on $E \subset \mathbb{C}$ is a function that assigns to each $z \in E$ one value from $f(z)$.

§2.1 Sometimes, mapping is not what you want so!

For any function, $f(z) : D \subseteq \mathbb{C} \to \mathbb{C}$ we can understand the function examining the outputs. Like, if we consider the inputs $z = x + iy$ then the output $w \in \mathbb{C}$ is also a complex number and we can say the complex function $f(z)$ is a pair of real functions.

$$\begin{aligned} w &= f(z) \\ &= f(x + iy) \\ &= u + iv \\ &= u(x, y) + iv(x, y) \end{aligned}$$

That means rather than understanding the complex function $f(z)$, we can understand the real functions $u(x, y)$ and $v(x, y)$. Similarly, in polar form $z = re^{i\theta}$ we get,

$$\begin{aligned} w &= f(z) \\ &= f(re^{i\theta}) \\ &= u + iv \\ &= u(r, \theta) + iv(r, \theta) \end{aligned}$$

Which one we need to use depends on the function you have given. Okay, now we will see how $\exp(z)$, $\log(z)$ and z^2 function work.

§2.1.i Complex Exponential

The exponential function,

$$\exp : \mathbb{C} \to \mathbb{C} - \{0\}$$

is defined to be

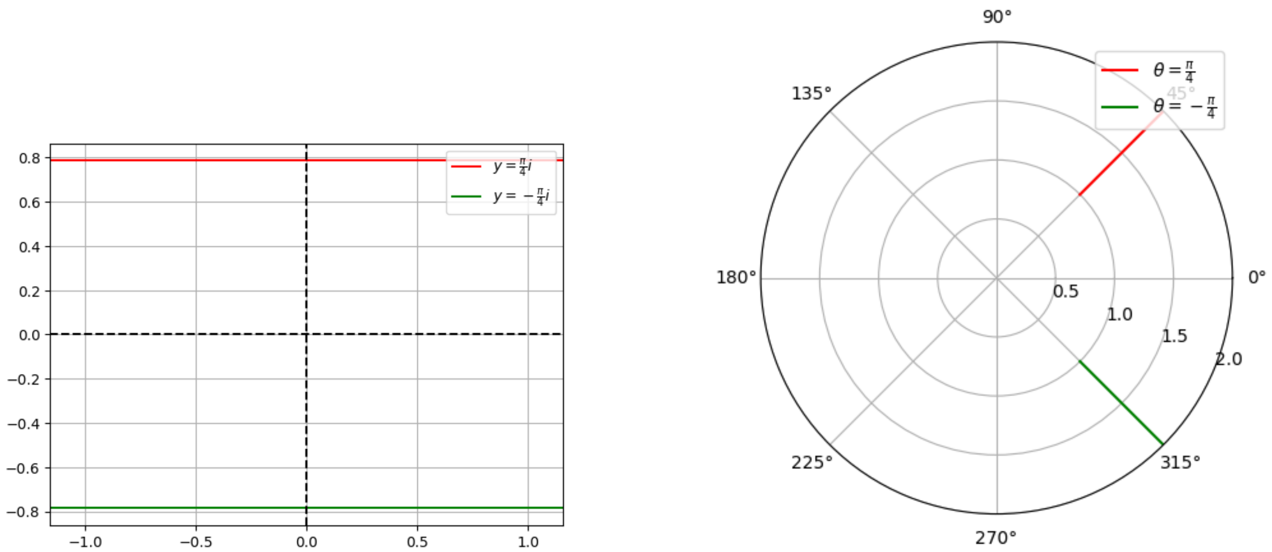
$$\exp(z) = \exp(x + iy) = e^x e^{iy}$$

Wait, isn't $e^{iy} = \cos(y) + i\sin(y)$? That means the output is a complex number which is naturally represented in a polar form whose modulus and argument are,

$$|\exp(x + iy)| = e^x \qquad \arg(\exp(x + iy)) = y + 2\pi\mathbb{Z}$$

> Can you see why the exponential map is a homomorphism from the addition group $(\mathbb{C}, +)$ to the multiplicative group $(\mathbb{C} - \{0\}, \cdot)$? From this perspective, we can say,
>
> $$\exp : (\mathbb{C}//2\pi\mathbb{Z}, +) \to (\mathbb{C} - \{0\}, \cdot)$$

How we can visualize this map? By using the $z-w$ plane. Let's see how this map acts for horizontal and vertical lines from z-plane to the w-plane. For $z = x + \frac{\pi}{4}i$ line (red), if we apply the complex exponent then we get,



(a) $z = x \pm \frac{\pi}{4}i$

(b) $r = e^x, \theta = \pm\frac{\pi}{4}$

Figure 2.1: Input and Output under the Exponential Map for Horizontal Lines

$$\begin{aligned} \exp z &= \exp\left(x + \frac{\pi}{4}i\right) \\ &= e^x e^{i\frac{\pi}{4}} \end{aligned}$$

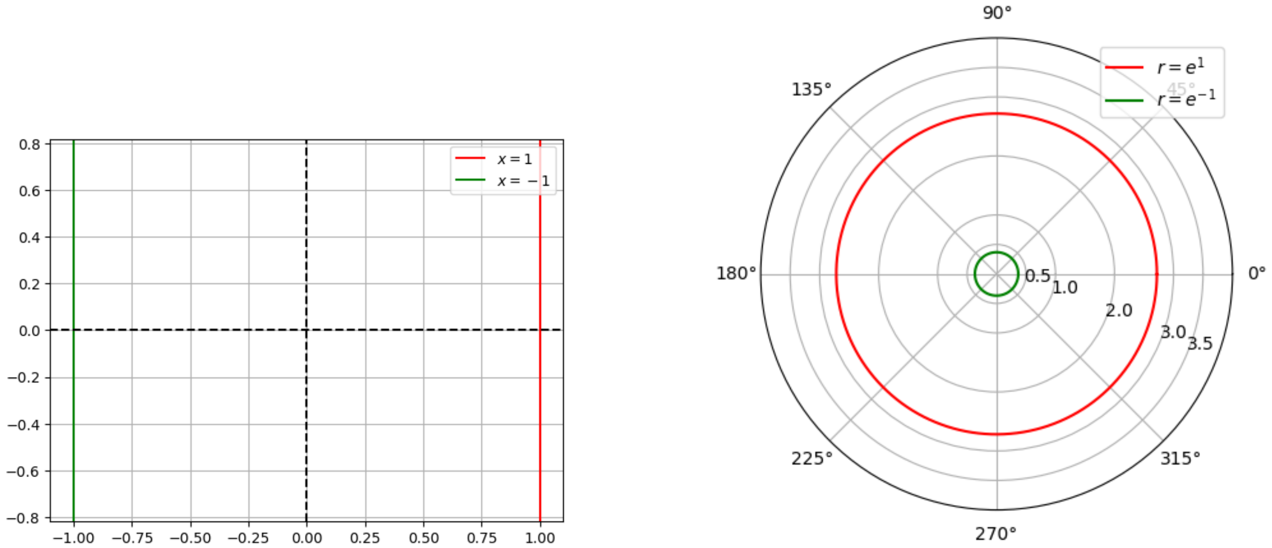
Here, if we compare the final result with the complex polar form then we get $r = e^x$ and $\theta = \frac{\pi}{4}$.

> Wait, why not we use the rectangular form? Because $e^x e^{i\frac{\pi}{4}} = e^x \cos\frac{\pi}{4} + ie^x \sin\frac{\pi}{4} = u(x) + iv(x)$ is not a good choice for here, as it needs to be considered both real and imaginary part in term of a varying x (as a function of x).

That means our arguments are fixed but radii are changing. Hence, get the red line which is emitting from the origin with angle $\frac{\pi}{4}$. But here is a gotcha! why this ray is not coming from exactly the origin? Can you guess that? Similarly, we can have

$$\begin{aligned}\exp z &= \exp\left(x - \frac{\pi}{4}i\right) \\ &= e^x e^{-i\frac{\pi}{4}}\end{aligned}$$

Okay, let's repeat the whole thing, but this time use the vertical lines in our input plane.



(a) $z = \pm 1 + iy$ (b) $r = e^{\pm 1}$

Figure 2.2: Input and Output under the Exponential Map for Vertical Lines

For example, if we take the vertical line $z = 1 + iy$ whose x coordinate is fixed and y can vary then we have,

$$\begin{aligned}\exp z &= \exp(1 + iy) \\ &= e^1 e^{iy}\end{aligned}$$

Again, if we compare the final result with the complex polar form then we get $r = e^1$ (fixed radius) and $\theta = y$ (which is changing). Hence, the end result is a circle with radius $r = e^1$ (red). I hope you can do the same for $z = -1 + iy$. Let's make things spicy! Now, we will see how the region between two vertical lines mapped in the output plane. The vertical strip $-1 \leq \mathrm{Re}(z) \leq 1$ in the z-plane maps to the annular region bounded by the circles $r = e^{-1}$ and $r = e^1$ in the w-plane under the exponential map $w = \exp(z)$, let's analyze how the exponential map transforms each component of $z = x + iy$. The range of x in the vertical strip is given by $-1 \leq x \leq 1$.
When $x = 1$, we have:

$$|w| = e^1 = e.$$

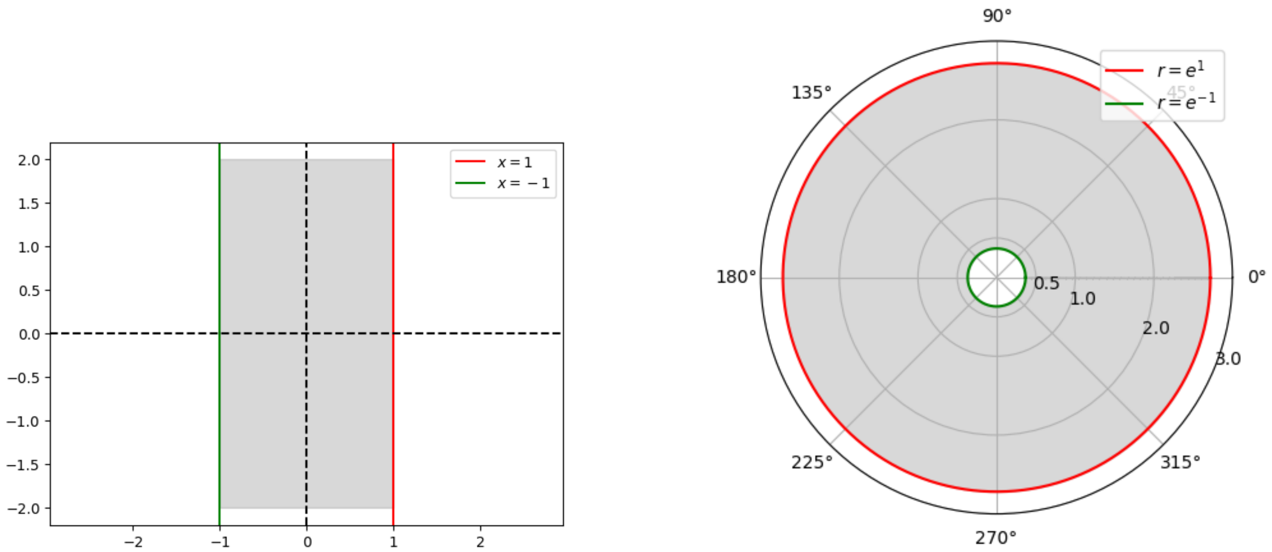
So, points on the line $x = 1$ in the z-plane are mapped to points on the circle of radius $r = e$ in the w-plane.
When $x = -1$, we have:

$$|w| = e^{-1} = \frac{1}{e}.$$

Thus, points on the line $x = -1$ in the z-plane are mapped to points on the circle of radius $r = \frac{1}{e}$ in the w-plane.

Since x varies continuously from -1 to 1, $|w| = e^x$ will take all values between e^{-1} and e. Therefore, the modulus $|w|$ in the w-plane lies in the range:

$$e^{-1} \leq |w| \leq e.$$



(a) $z = \pm 1 + iy$ (b) $r = e^{\pm 1}$

Figure 2.3: Input and Output under the Exponential Map for Vertical Strip

This defines an annular region bounded by the circles $r = e^{-1}$ and $r = e$ in the w-plane. The imaginary part y of z becomes the angle $\arg(w)$ in the w-plane. Since y can vary freely from $-\infty$ to $+\infty$, $\arg(w)$ will cover all possible angles from $-\infty$ to $+\infty$. This means that w wraps around the annular region **infinitely many times** as y varies, covering every point in the annulus.

Exercise 2.1. Find the output region for the horizontal strip in z-plane we mentioned above (between $z = x + \frac{\pi}{4}i$ and $z = x - \frac{\pi}{4}i$).

§2.1.ii Complex powers

With the logarithm function at our disposal, we are able to define complex powers of complex numbers. Let α be a complex number. The for all $z \neq 0$, we define the α-th power z^α of z by

$$\boxed{z^\alpha \equiv e^{\alpha \log(z)} = e^{\alpha \log|z| + i\alpha \arg(z)}}$$

The multiple-valuedness of the argument means that generically there will be an infinite number of values for z^α. We can rewrite the above expression a little to make this manifest:

$$z^\alpha = e^{\alpha \log|z| + i\alpha \operatorname{Arg}(z) + i\alpha 2\pi k} = e^{\alpha \log(z)} e^{i\alpha 2\pi k},$$

for $k = 0, \pm 1, \pm 2, \ldots$ Depending on α we will have either one, finitely many or infinitely many values of $\exp(i2\pi\alpha k)$.

Exercise 2.2. Find i^i and 1^i.

§2.2 Complex Trigonometry! Ugh!

Before we start talking about complex trigonometric functions, let's look at the hyperbolic functions. We already knew that,

$$\sinh x = \frac{e^x - e^{-x}}{2}$$
$$\cosh x = \frac{e^x + e^{-x}}{2}$$
$$\tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

Okay, now without worrying about anything just plug in the complex numbers (purely imaginary) in the trigonometric function!

$$\begin{aligned}\cos(ix) &= \frac{e^{i(ix)} + e^{-i(ix)}}{2} \\ &= \frac{e^{-x} + e^{x}}{2} \\ &= \frac{e^{x} + e^{-x}}{2} \\ &= \cosh x\end{aligned}$$

In a similar fashion, we can get

$$\begin{aligned}\sin ix &= i \sinh x \\ \tan ix &= i \tanh x\end{aligned}$$

Nice! But we need to plug in the general complex number ($z = a + ib$), so let's give it a try also,

$$\begin{aligned}\sin(x + iy) &= \sin(x)\cos(iy) + \cos(x)\sin(iy) \\ &= \sin(x)\cosh(y) + \cos(x) i \sinh(y) \\ &= \sin(x)\cosh(y) + i\cos(x)\sinh(y)\end{aligned}$$

Question: Do we have the same properties (nicer) of trigonometric function in complex variables?

Example 2.1

Similarly, we can show that,

$$\cos z = \cos(x + iy) = \cos x \cos(iy) - \sin x \sin(iy) = \cos x \cosh y - i \sin x \sinh y$$

Example 2.2

Question: Show that,

$$\sin(-z) = \sin(z)$$

Example 2.3

Question: Show that,

$$\sin(z + 2\pi) = \sin(z)$$

Example 2.4

Question: Solve for $\sin(z) = 2$ where $z \in \mathbb{C}$.
Answer: Since

$$\sin(z) = \sin(x)\cosh(y) + i\cos(x)\sinh(y)$$

we need

$$\sin(x)\cosh(y) = 2 \text{ and } \cos(x)\sinh(y) = 0$$

simultaneously. The second equation is eeasier to work with because we can use the zero product principle. It gives:

$$x = \pi/2 + k\pi \text{ or } y = 0$$

We substitute these results into the first equation one at at time. If $y = 0$, the first equation becomes:

$$\sin(x)\cosh 0 = 2$$

Since $\cosh(0) = 1$, this leads to the equation $\sin(x) = 2$ which has no solutions. Next, if $x = \pi/2 + k\pi$ then $\sin(x) = \pm 1$. Thus we consider cases. If where k is even, then

$$\sin(x) = \sin\left(\frac{\pi}{2} + k\pi\right) = 1$$

and we arrive at

$$\cosh(y) = 2$$

which has two solutions (verify)

$$y = \ln(2 \pm \sqrt{3})$$

Finally, if $x = \pi/2 + k\pi$ where k is odd, then

$$\sin(x) = \sin\left(\frac{\pi}{2} + k\pi\right) = -1$$

and we arrive at

$$\cosh(y) = -2$$

which has no solutions (verify). Thus the equation $\sin(z) = 2$ has as its solutions

$$z = \left[\frac{\pi}{2} + k\pi\right] + i[\ln(2 \pm \sqrt{3})]$$

where k is an even integer.

Example 2.5

Question: Find the image of the given line under the given map,

1. $Im(z) = 1; \quad f(z) = \cos(z)$
2. $Re(z) = \frac{\pi}{6}; \quad f(z) = \sin(z)$

Answer: (1) Using the formula for $\cos(z)$ in terms of exponential functions:

$$\cos(z) = \cos(x + i) = \frac{e^{i(x+i)} + e^{-i(x+i)}}{2}.$$

Simplify $e^{i(x+i)}$ and $e^{-i(x+i)}$:

$$e^{i(x+i)} = e^{-1}e^{ix}, \quad e^{-i(x+i)} = e^{-1}e^{-ix}.$$

Thus:

$$\cos(z) = \frac{e^{-1}e^{ix} + e^{-1}e^{-ix}}{2} = e^{-1}\frac{e^{ix} + e^{-ix}}{2}.$$

Recognizing $\frac{e^{ix}+e^{-ix}}{2} = \cos(x)$:

$$\cos(z) = e^{-1}\cos(x).$$

Since $x \in \mathbb{R}$, $\cos(x)$ takes values in $[-1, 1]$. Therefore: $f(z) = \cos(z)$ maps the line $\text{Im}(z) = 1$ to the horizontal segment $\left[-\frac{1}{e}, \frac{1}{e}\right]$ on the real axis.

(2) If $\text{Re}(z) = \frac{\pi}{6}$, we can write $z = \frac{\pi}{6} + iy$, where $y \in \mathbb{R}$.

Using the formula for $\sin(z)$ we get:

$$\sin\left(\frac{\pi}{6} + iy\right) = \sin\left(\frac{\pi}{6}\right)\cosh(y) + i\cos\left(\frac{\pi}{6}\right)\sinh(y)$$
$$= \underbrace{\frac{1}{2}\cosh(y)}_{u} + i\underbrace{\frac{\sqrt{3}}{2}\sinh(y)}_{v}$$

Now, we want a relation between u and v in order to get equation for w-plane. We knew that,

$$\cosh^2(y) - \sinh^2(y) = 1.$$

Substitute the expressions for u and v:

$$(2u)^2 - \left(\frac{2}{\sqrt{3}}v\right)^2 = 1.$$

The relationship between u and v is a hyperbola. This shows that the image of the line $\mathrm{Re}(z) = \frac{\pi}{6}$ under $f(z) = \sin(z)$ is a branch of a hyperbola in the w-plane.

3 Limit

Example 3.1

Question: Find

$$\lim_{z\to e^{\pi i/3}} (z - e^{\pi i/3}) \frac{z}{z^3+1}$$

Answer: We knew that, $e^{i\pi} = -1$ which implies $z^3 + 1 = e^{i\pi} + 1 = -1 + 1 = 0$. That means direct substitution will produce $\frac{0}{0}$. So, we can apply L'Hopital rules here.

$$\begin{aligned}
\lim_{z\to e^{\pi i/3}} (z - e^{\pi i/3}) \frac{z}{z^3+1} &= \lim_{z\to e^{\pi i/3}} \frac{z^2 - e^{\pi i/3} z}{z^3+1} \\
&= \lim_{z\to e^{\pi i/3}} \frac{(2z - e^{\pi i/3})}{3z^2} \\
&= \frac{e^{\pi i/3}}{3e^{2\pi i/3}} \\
&= \frac{1}{3} e^{-\pi i/3}
\end{aligned}$$

Example 3.2

Question: Find

$$\lim_{z\to 0} \left(\frac{\sin z}{z}\right)^{\frac{1}{z^2}}$$

Answer: Let

$$\begin{aligned}
w &= \lim_{z\to 0} \left(\frac{\sin z}{z}\right)^{\frac{1}{z^2}} \\
\ln w &= \ln \left[\lim_{z\to 0} \left(\frac{\sin z}{z}\right)^{\frac{1}{z^2}}\right] \\
&= \lim_{z\to 0} \left[\ln \left(\frac{\sin z}{z}\right)^{\frac{1}{z^2}}\right] \\
&= \lim_{z\to 0} \frac{1}{z^2} \ln \left(\frac{\sin z}{z}\right) \\
&= \lim_{z\to 0} \frac{\ln\left(\frac{\sin z}{z}\right)}{z^2} \\
&= \lim_{z\to 0} \frac{\ln(\sin z) - \ln z}{z^2} \\
&\overset{\text{LH}}{=} \lim_{z\to 0} \frac{\frac{1}{\sin z}\cos z - \frac{1}{z}}{2z} \\
&= \lim_{z\to 0} \frac{z\cos z - \sin z}{2z^2 \sin z}
\end{aligned}$$

$$
\begin{aligned}
\ln w &\overset{\text{LH}}{=} \lim_{z\to 0} \frac{\cos z + z\cdot(-\sin z) - \cos z}{4z\sin z + 2z^2\cos z} \\
&= \lim_{z\to 0} \frac{-z\sin z}{4z\sin z + 2z^2\cos z} \\
&\overset{\text{LH}}{=} \lim_{z\to 0} \frac{-\sin z - z\cos z}{4\sin z + 8z\cos z - 2z^2\sin z} \\
&\overset{\text{LH}}{=} \lim_{z\to 0} \frac{-\cos z - \cos z - z(-z\sin z)}{\cdots} \\
&= \frac{-1-1}{4+8} \\
&= -\frac{1}{6}
\end{aligned}
$$

Example 3.3

Show that the limit does not exist,

$$\lim_{z\to 0} \frac{\bar{z}}{z}$$

Example 3.4

Show that the limit does not exist,

$$\lim_{z\to 0} \frac{xy}{x^2+y^2}$$

Example 3.5

Question: Find all the discontinuous points for $f(z) = \cot(z)$.
Answer:

$$
\begin{aligned}
f(z) &= \cot(z) \\
&= \frac{\cos z}{\sin z}
\end{aligned}
$$

Discontinuous at $\sin z = 0$ which implies,

$$
\begin{aligned}
\frac{e^{iz} - e^{-iz}}{2i} &= 0 \\
e^{iz} &= e^{-iz} \\
\frac{e^{iz}}{e^{-iz}} &= 1 \\
e^{2iz} &= 1 \\
e^{2iz} &= e^{i(0+2\pi k)} \\
2z &= 2\pi k \\
z &= k\pi, k \in \mathbb{Z}
\end{aligned}
$$

§3.1 Exercise

Exercise 3.1. Let $f(z) = \begin{cases} \frac{z^2+4}{z-2i}, z \neq 2i \\ 3+4i, z = 2i \end{cases}$. Is $f(z)$ continuous at $z = 2i$?

Exercise 3.2. Find all points of discontinuity for the function,

$$f(z) = \frac{2z-3}{z^2+2z+2}$$

Exercise 3.3. Find the following limits,

$$\lim_{z\to 0} \frac{z-\sin(z)}{z^3}, \lim_{z\to 0} \frac{\tan(z)-\sin(z)}{z^3}$$

Exercise 3.4. If $f(z) = \begin{cases} \frac{z^2-4}{z^2-3z+2}, z \neq 2 \\ kz^2+6, z = 2 \end{cases}$, find k such that the function $f(z)$ becomes continuous at $z = 2$.

4 Differentiation

$\mathcal{N}(x)$ will denote the neighborhood at point x.

Definition 4.1 (Geometric)**.** The derivative is the slope of a line tangent to the graph of the function, if the graph has a tangent.

Definition 4.2 (Approximation)**.** The derivative of a function is the best linear approximation to the function near a point.

Definition 4.3 (Infinitesimal)**.** The ratio of the infinitesimal change in the value of a function to the infinitesimal change in a function.

If we start out with $f : \mathbb{R} \to \mathbb{R}$, then $f'(c)$ is an approximation of how f changes in a small interval around $x = c$. For example, let $f(x) = x^2$, and $c = 2$. Then $f'(2) = 4$. Notice that $f(1.01) = 1.0201$. Then the change from $f(1)$ to $f(1.01)$ is 0.0201. This is approximately $2(.01) = 0.02$.

For higher dimensions, the derivative needs to be a transformation between $\mathbb{R}^n$ and $\mathbb{R}^m$. For example, take $f(x, y) = (x^2, y^2)$. Then the derivative is

$$J_{ij} = \frac{\partial f_i}{\partial x_j} = \begin{pmatrix} \nabla^T f_1 \\ \nabla^T f_2 \end{pmatrix} = \begin{pmatrix} 2x & 0 \\ 0 & 2y \end{pmatrix}.$$

At $(x, y) = (1, 1)$ this is

$$\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}.$$

Moving from $f(1, 1) = (1, 1)$ to $f(1.01, 1.01) = (1.0201, 1.0201)$. The change between the function values is $(0.0201, 0.0201)$. Notice that

$$\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 0.01 \\ 0.01 \end{pmatrix} = \begin{pmatrix} 0.02 \\ 0.02 \end{pmatrix}.$$

Again, very close. So, the derivative is the linear transformation that most closely fits the function. Since linear transformations are much easier to study than functions in general, we may learn a lot about the function from its derivatives.

One approach is to use the fact the "differentiability" is equivalent to "approximate linearity", in the sense that if f is defined in some neighborhood of a, then

$$f'(a) = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h} \quad \text{exists}$$

if and only if

$$f(a + h) = f(a) + f'(a)h + o(h) \quad \text{at } a \text{ (i.e., "for small } h\text{")}.$$

Before delving deeply into the complex realm, let's grasp the most essential tool for comprehending a complex plane (complex vector space, complex manifold): the holomorphic function. A common query might arise: If we consider points in the complex plane akin to vectors in $\mathbb{R}^2$, does differentiability in $\mathbb{R}^2$ extend to the complex plane as well? The answer is no; complex differentiability is significantly more constrained. For if a function $f : \mathbb{R}^2 \to \mathbb{R}^2$ is differentiable at a point $x \in \mathbb{R}^2$ we can locally approximate it by a linear map that's why we have,

$$f(x + h) = f(x) + Ah + o(h), \qquad h \to 0$$

where $A : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation. Now we can do the same thing for a complex linear map,

$$f(x+h) = f(x) + \tilde{a} \cdot h + o(h), \qquad h \to 0$$

where $\tilde{a} \in \mathbb{C}$. Then we can see the complex multiplication $\tilde{a} \cdot h$ as a complex linear map which can be represented by a matrix form because we consider $\tilde{a}$ as a mapping $\mathbb{R}^2 \to \mathbb{R}^2$,

$$\underbrace{\begin{pmatrix} \operatorname{Re}\tilde{a} & -\operatorname{Im}\tilde{a} \\ \operatorname{Im}\tilde{a} & \operatorname{Re}\tilde{a} \end{pmatrix}}_{\tilde{a}} \underbrace{\begin{pmatrix} \operatorname{Re}h \\ \operatorname{Im}h \end{pmatrix}}_{h} = \begin{pmatrix} \operatorname{Re}\tilde{a}\operatorname{Re}h & -\operatorname{Im}\tilde{a}\operatorname{Im}h \\ \operatorname{Im}\tilde{a}\operatorname{Re}h & \operatorname{Re}\tilde{a}\operatorname{Im}h \end{pmatrix}$$

This encodes the complex multiplication $\tilde{a} \cdot h = (\operatorname{Re}\tilde{a} + \operatorname{Im}\tilde{a}) \cdot (\operatorname{Re}h + \operatorname{Im}h) = (\operatorname{Re}\tilde{a}\operatorname{Re}h - \operatorname{Im}\tilde{a}\operatorname{Im}h) + i(\operatorname{Im}\tilde{a}\operatorname{Re}h + \operatorname{Re}\tilde{a}\operatorname{Im}h)$, but not all linear maps which arise from $\mathbb{R}$-differentiable function look like this.

A function $f(z)$ is said to be analytic or holomorphic at z_0 if one of the following equivalent conditions holds:

C1 We can consider f as a function of two real variables $f(x, y)$. And we can decompose it as $f(x, y) = u(x, y) + iv(x, y)$, where $u(x, y)$ and $v(x, y)$ denote the real and imaginary part of f respectively. Then we want the first partial derivatives to exist and be continuous and further satisfy the CR equations,

$$u_x = v_y, \quad v_x = -u_y$$

C2 The limit exists

$$\lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z} \quad \forall z \in \mathcal{N}(z_0)$$

C3 $\exists$ a power series of the form $\sum a_n(z - z_0)^n$ which is convergent to $f(z)$ for each point z in a neighborhood of z_0.

Example 4.1

Question: Show that $f(z) = \operatorname{Re}(z)$ is nowhere differentiable.
Answer: We know that a function is differentiable at point z_0 if the following limit exists,

$$\lim_{h \to 0} \frac{f(z_0 + h) - f(z_0)}{h}$$

Remember, both z_0 and h are complex numbers. Let's choose an arbitrary point $z \in \mathbb{C}$ and substitute $z = x + iy$ and $h = h_x + ih_y$ then,

$$\begin{aligned} \frac{f(z+h) - f(z)}{h} &= \frac{\operatorname{Re}(z+h) - \operatorname{Re}(z)}{h} \\ &= \frac{\operatorname{Re}(x + iy + h_x + iy) - \operatorname{Re}(x + iy)}{h} \\ &= \frac{\operatorname{Re}((x + h_x) + i(y + h_y)) - \operatorname{Re}(x + iy)}{h} \\ &= \frac{x + h_x - x}{h} \\ &= \frac{h_x}{h} \end{aligned}$$

Now, we will consider different different paths and show they give different different values.
Path 1 - Along the real axis
If we approach the origin along the real axis then we have, $h = h_x + i \cdot 0$ which implies $\operatorname{Re}(h) =$

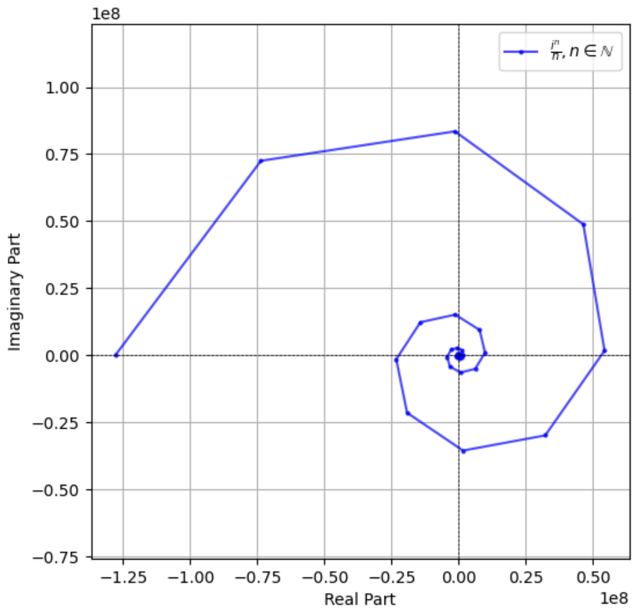


Figure 4.1: $\frac{i^n}{n}$

$h_x \equiv h = h_x + i \cdot 0$. Hence,

$$\begin{aligned}\lim_{h\to 0} \frac{h_x}{h} &= \lim_{h_x\to 0} \frac{h_x}{h_x + i \cdot 0} \\ &= \lim_{h_x\to 0} \frac{h_x}{h_x} \\ &= \lim_{h_x\to 0} 1 \\ &= 1\end{aligned}$$

Path 1 - Along the imaginary axis
If we approach the origin along the imaginary axis then we have, $h = 0 + i \cdot h_y$ which implies $\text{Re}(h) = 0$. Hence,

$$\begin{aligned}\lim_{h\to 0} \frac{h_x}{h} &= \lim_{h_y\to 0} \frac{h_x}{0 + ih_y} \\ &= \lim_{h_y\to 0} \frac{0}{ih_y} \\ &= 0\end{aligned}$$

Voila! We show that the limit doesn't exist when we choose two different paths for which $h \to 0$.

Alternative: While choosing different paths to solve this kind of limit problem, getting the same values for two paths might be very problematic. Hence, we can solve this problem by choosing a special path. Like $h_n = \frac{i^n}{n}$ shown in figure 4.1. Now, we have

$$\begin{aligned}\lim_{h_n\to 0} \frac{\text{Re}(h_n)}{h_n} &= \lim_{h_n\to 0} \frac{\frac{\text{Re}(i^n)}{n}}{\frac{i^n}{n}} \\ &= \begin{cases} 1, n \text{ is even} \\ 0, n \text{ is odd} \end{cases}\end{aligned}$$

The end result depends on the value of n. Hence, the limit doesn't exist.

Example 4.2

Question: Prove that $f(z) = |z|^2$ is differentiable only at the origin.
Answer: Pick any arbitrary point $z_0 \in \mathbb{C}$. Then check the differentiability at that point. Besides, we knew that, $|z|^2 = z\bar{z}$.

$$\begin{aligned}
\lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} &= \lim_{\Delta z \to 0} \frac{|\, z_0 + \Delta z \,|^2 - |z_0|^2}{\Delta z} \\
&= \lim_{\Delta z \to 0} \frac{(z_0 + \Delta z)(\overline{z_0 + \Delta z}) - z_0\overline{z_0}}{\Delta z} \\
&= \lim_{\Delta z \to 0} \frac{z_0\overline{z_0} + z_0\overline{\Delta z} + \Delta z\overline{z_0} + \Delta z\overline{\Delta z} - z_0\overline{z_0}}{\Delta z} \\
&= \lim_{\Delta z \to 0} z_0\frac{\overline{\Delta z}}{\Delta z} + \overline{z_0} + \overline{\Delta z}
\end{aligned}$$

Here, we can't substitute $\Delta z = 0$ in the expression. Hence, to find the limit value we need to consider pathwise value.
Path-I: Consider the x-direction:
Here, $y = 0$ hence $\Delta y = 0$. And we know that, $\Delta z = \Delta x + i\Delta y = \Delta x$. Even $\overline{\Delta z} = \Delta x - i\Delta y = \Delta x$. And $\Delta z \to 0 \implies \Delta x \to 0$. Now, using all these information rewrite the limit expression:

$$\begin{aligned}
\lim_{\Delta z \to 0} z_0\frac{\overline{\Delta z}}{\Delta z} + \overline{z_0} + \overline{\Delta z} &= \lim_{\Delta x \to 0} z_0\frac{\Delta x}{\Delta x} + \overline{z_0} + \Delta x \\
&= \lim_{\Delta x \to 0} z_0 + \overline{z_0} + \Delta x \\
&= \boxed{z_0 \cdot 1 + \overline{z_0} + 0}
\end{aligned}$$

Similarly, For **Path-II: Consider the y-direction:**
Here, $x = 0$ hence $\Delta x = 0$. And we know that, $\Delta z = \Delta x + i\Delta y = i\Delta y$. Even $\overline{\Delta z} = \Delta x - i\Delta y = -i\Delta y$. And $\Delta z \to 0 \implies \Delta y \to 0$. Now, using all these information rewrite the limit expression:

$$\begin{aligned}
\lim_{\Delta z \to 0} z_0\frac{\overline{\Delta z}}{\Delta z} + \overline{z_0} + \overline{\Delta z} &= \lim_{\Delta y \to 0} z_0\frac{i\Delta y}{-i\Delta y} + \overline{z_0} + \Delta y \\
&= \lim_{\Delta y \to 0} z_0 \cdot -1 + \overline{z_0} + \Delta y \\
&= \boxed{-z_0 + \overline{z_0} + 0}
\end{aligned}$$

Equating the limit values we get:

$$\begin{aligned}
z_0 + \overline{z_0} &= z_0 + \overline{z_0} \\
2z_0 &= 0 \\
z_0 &= 0
\end{aligned}$$

Which mean the function is only differentiable at point $z = 0$. Except that point the function is not differentiable.

Example 4.3

Question: Determine a harmonic conjugate to the function

$$f(x, y) = 2y^3 - 6x^2y + 4x^2 - 7xy - 4y^2 + 3x + 4y - 4$$

We first of all check if $f(x, y)$ is indeed a harmonic function. This amounts to show $f(x, y)$

satisfy the two-dimensional Laplace equation

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0 \tag{1}$$

We have $\frac{\partial^2 f}{\partial x^2} = 8 - 12y$ and $\frac{\partial^2 f}{\partial y^2} = 12y - 8$. Thus, (1) is fulfilled, and so $f(x, y)$ is harmonic.

Next, we seek to determine a harmonic conjugate to the given function. Let $u(x, y) = 2y^3 - 6x^2y + 4x^2 - 7xy - 4y^2 + 3x + 4y - 4$.

$$u_x = v_y \iff -12xy + 8x - 7y + 3 = v_y$$

Integrate with respect to y

$$v = -6xy^2 + 8xy - \frac{7}{2}y^2 + 3y + h(x) \tag{2}$$

where $h(x)$ is a function of x alone. To determine this, we use the second Cauchy-Riemann equation* $v_x = -u_y$

$$\begin{aligned} -u_y = v_x &\iff 6x^2 + 7x - 6y^2 + 8y - 4 = h'(x) - 6y^2 + 8y \\ &\iff h'(x) = 6x^2 + 7x - 4 \end{aligned}$$

Integrating with respect to x we have

$$h(x) = 2x^3 + \frac{7}{2}x^2 - 4x + C$$

where C is an arbitrary constant. Therefore, if we let $C = 0$, then one harmonic conjugate of u is given as:

$$v = 2x^3 + \frac{7}{2}x^2 - 6xy^2 + 8xy - 4x - \frac{7}{2}y^2 + 3y$$

Yet another shortcut. Since u is harmonic (on the simply connected domain $\mathbb{C}$), there has to be a harmonic conjugate v. Let $F = u + iv$ be the corresponding holomorphic function. It follows from (the derivation of) Cauchy Riemann's equations that:

$$F' = u'_x - i\, u'_y = -12xy + 8x - 7y + 3 + i(6x^2 + 7x - 6y^2 + 8y - 4).$$

Let $G(z) = 3 + 8z + i(6z^2 + 7z - 4)$. Then $G(z) = F'(z)$ if z is real, so by the identity theorem, $G = F'$ for all z. Hence

$$F(z) = 3z + 4z^2 - 4 + i(2z^3 + \frac{7}{2}z^2 - 4z + C)$$

for some real constant C (the real part of the constant of integration has to be 4 to match u). Finally

$$v = \mathrm{Im}(F(z)).$$

5 Line Integral

- If you can easily parameterize your curve C by $\gamma(t)$ then use:

$$\int_C f(z)dz \overset{C=\gamma(t)}{=\!=} \int_\gamma f(\gamma(t))\gamma'(t)dt$$

For example, if our curve is a circle $|z-z_0| = R$ then we can parametrize it by $\gamma(t) = z_0+Re^{it}, 0 \geq t \geq 2\pi$.

- If you can deform your parametrization from initial path $\gamma_1(t)$ to final path $\gamma_2(t)$ without hitting any singularities then:

$$\int_{\gamma_1} f(z)dz = \int_{\gamma_2} f(z)dz$$

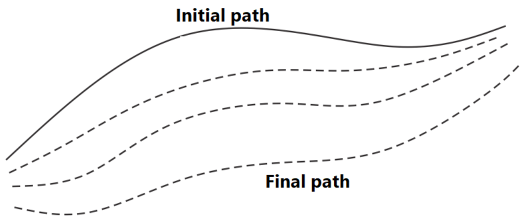


Figure 5.1: Path deformation: while deforming the initial path, we didn't hit any singularities

- (**Cauchy's theorem**) If the curve is closed and our function $f(z)$ is analytic on the region D enclosed by the curve then,

$$\int_\gamma f(z)dz = 0$$

- (**Cauchy's integral formula**) Suppose C is a simple closed curve and the function $f(z)$ is analytic on the region D closed by the curve. We assume C is oriented counterclockwise. Then for any z_0 inside C:

$$f(z_0) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-z_0}dz$$

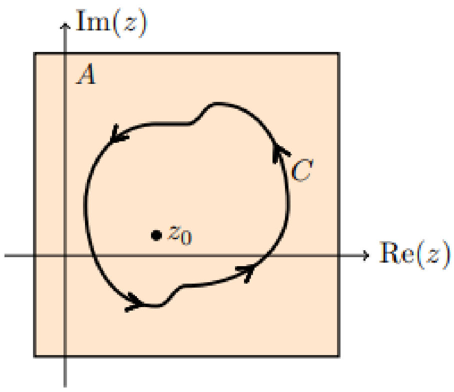


Figure 5.2: Cauchy integral formula

- (**Cauchy's integral formula for derivatives**) If $f(z)$ and C satisfy the same hypotheses as for Cauchy's integral formula then, for all z_0 inside C we have:

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \int_C \frac{f(z)}{(z - z_0)^{n+1}} dz$$

- (**Cauchy's residue theorem**) Suppose $f(z)$ is analytic in the region A except for a set of isolated singularities. Also, suppose C is a simple closed curve in A that doesn't go through any of the singularities of f and is oriented counterclockwise. Then

$$\int_C f(z)dz = 2\pi i \sum \text{ residues of } f \text{ inside } C$$

To calculate residue at the pole z_0 with degree k, we use:

$$\text{Res}(f, z_0) = \lim_{z \to z_0} \frac{1}{(k-1)!} \frac{d^{k-1}}{dz^{k-1}} \left[(z - z_0)^k f(z)\right]$$

But we can compute it easily using a series in our function if possible.

6 Laplace Transformation

§6.1 Derivation

We knew that,

$$\begin{aligned}
F(s) &= \int_0^\infty f(t)e^{-st}dt \\
\frac{d}{ds}F(s) &= \frac{d}{ds}\int_0^\infty f(t)e^{-st}dt \\
&\overset{\text{Leibnitz}}{=} \int_0^\infty f(t)\frac{\partial}{\partial s}e^{-st}dt \qquad \text{(also use Dominated convergence theorem)} \\
&= -\int_0^\infty tf(t)e^{-st}dt \\
-\frac{d}{ds}F(s) &= \mathcal{L}\{tf(t)\} \quad \blacksquare
\end{aligned}$$

Again, we use Fubini's theorem to derive the integral formula,

$$\int_s^\infty F(s) = \int_0^\infty f(t)\left(\int_s^\infty e^{-ut}du\right)dt$$

We can categorize all the examples into 6 types.

1. Known formula
2. $$\mathcal{L}\{e^{at}f(t)\} = \mathcal{L}\{f(t)\}_{s\to s-a} = F(s-a)$$
$$\underbrace{\mathcal{L}^{-1}\{F(s)\}}_{\text{unknown}} = e^{at}\underbrace{\mathcal{L}^{-1}\{F(s+a)\}}_{\text{known}}$$
3. $$\mathcal{L}\{f(t)u(t-a)\} = \mathcal{L}\{f(t+a)\}e^{-as}$$
$$\mathcal{L}^{-1}\{F(s)e^{-as}\} = f(t-a)u(t-a)$$
4. Partial fraction type
5. s-differentiation
6. Convolution type

§6.2 Type-5

Question: Find the inverse Laplace Transformation of $\frac{2s}{(s^2+1)^2}$.
Answer: Use the formula,

$$\mathcal{L}\{tf(t)\} = -\frac{d}{ds}\mathcal{L}\{f(t)\}$$

We already know that,

$$\mathcal{L}\{\sin(t)\} = \frac{1}{s^2+1}$$

Then,

$$\begin{aligned}
\mathcal{L}\{t\sin t\} &= -\frac{d}{ds}\left(\frac{1}{s^2+1}\right) \\
&= \frac{2s}{(s^2+1)^2}
\end{aligned}$$

§6.3 Type-6

Question: Find the Laplace Transformation of $\frac{1}{s^2}\frac{1}{s+1}$.
Answer: We already knew that,

$$\mathcal{L}\left\{\frac{1}{s^2}\right\} = t\,u(t), \qquad \mathcal{L}\left\{\frac{1}{s^2+1}\right\} = \sin t\,u(t)$$

Now, use the convolution formula,

$$\begin{aligned}\mathcal{L}^{-1}\left\{\frac{1}{s^2}\frac{1}{s^2+1}\right\} &= \int_{-\infty}^{\infty} f(t-\tau)g(\tau)d\tau \\ &= \int_{-\infty}^{\infty} (t-\tau)\,u(t-\tau)\sin(\tau)\,u(\tau)d\tau \\ &\overset{A}{=} \int_0^t (t-\tau)\sin\tau d\tau\end{aligned}$$

See the footnote for A[1].

[1] $u(t-\tau)$ force $t-\tau \geq 0 \implies t \geq \tau$ and $u(\tau)$ force $\tau \geq 0$. Hence, $0 \leq \tau \leq t$

7 Chin Chapak Dam Dam

Definition 7.1 (qwerty). qwerty is qwertyasdf ...

§7.1 sec 1

Theorem 7.1 (pythagoras theorem)
$a^2 + b^2 = c^2$

Lemma 7.2 (einstein)
$E = mc^2$.

Corollary 7.3
$E = m\left(a^2 + b^2\right)$.

§7.2 sec 2

Proposition 7.4
asdfgh

Proof. this is the proof ■

Example 7.1
This is an example

There are some theoremstyles, which work outside boxes:

Example 7.1. sometimes examples are really large, in that case making it boxed doesnt look good in my opinion. I use this format of example in such a scenario.

Abuse of Notation. this is for abuse of notation, this works without numbering

Remark. this is a remark, this also works without numbering

Question. this is a question. does this work without numbering?

Exercise 7.1. this is an exercise

Problem 7.1. and this is a problem