MAT215: Complex Variables And Laplace Transformations

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LECTURE-01

Complex plane

As a set, the complex number field $\mathbb{C}$ be the set $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$. The set is a plane, so we call it the complex plane. To make it a field, we define addition and product:

$$(a,b) + (c,d) \stackrel{\text{def}}{=} (a+c, b+d)$$

$$(a,b)(c,d) \stackrel{\text{def}}{=} (ac - bd, bc + da)$$

Why do we define the operations in such a way? One answer is to mimic the complex operation. Another is to consider the multiplication as the map $z_1 \mapsto z_1 z_2$ (which is a real-linear operator).

$$a + ib = a\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + b\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$$

Problem

Show that the additive identity is $\mathbf{0} = (0,0)$, *the multiplicative identity is* $\mathbf{1} = (1,0)$ *and the multiplicative inverse* $(a,b)^{-1} = \left(\frac{a}{a^2+b^2}, \frac{-b}{a^2+b^2}\right)$

Complex numbers

Problem

Find the real numbers x and y such that $3x + 2iy - ix + 5y = 7 + 5i$

Problem

Prove that every real-linear operator on $\mathbb{C}$, that is every 2×2 real matrix M, can be represented by two complex numbers ξ and ζ and the formula $z \mapsto \xi z + \zeta \bar{z}$.

Hint: *Use $T(z) = T(x + iy)$*

Complex number

Given a complex number $z \stackrel{\text{def}}{=} x + iy$, its "evil twin" is the complex conjugate of z:

$$\bar{z} \stackrel{\text{def}}{=} x - iy$$

Find the real and imaginary parts.
We can show that,

$$x^3 + y^3 + 3ixy = \left(\frac{z+\bar{z}}{2}\right)^3 + \left(\frac{z-\bar{z}}{2i}\right)^3 + 3i\left(\frac{z+\bar{z}}{2}\right)\left(\frac{z-\bar{z}}{2i}\right)$$

Problem

Guess all the properties of Conjugate.

z and $\bar{z}$ are not independent variables, why?

Problem

Problem

Find the square root of $-15-8i$.

Problem

Solve the equation $z^2+(2i-3)z+5-i=0$.

Argument

The function $\mathbf{Arg}(z) : \mathbb{C}\backslash\{0\} \to (-\pi, \pi]$ is defined as follows:

$$\operatorname{Arg}(z) = \begin{cases} \arctan \frac{y}{x} & \text{if } x > 0, y \in \mathbb{R} \\ \arctan \frac{y}{x} + \pi & \text{if } x < 0, y \geq 0 \\ \arctan \frac{y}{x} - \pi & \text{if } x < 0, y < 0 \\ \frac{\pi}{2} & \text{if } x = 0, y > 0 \\ -\frac{\pi}{2} & \text{if } x = 0, y < 0 \\ \text{undefined} & \text{if } x = 0, y = 0 \end{cases}$$

Thus, if $z = r(\cos\Theta + i\sin\Theta)$, with $r > 0$ and $-\pi < \Theta \leq \pi$, then

$$\arg(z) = \{\operatorname{Arg}(z) + 2n\pi \mid n \in \mathbb{Z}\}.$$

Polar form

Properties of Modulus:

$$|z_1 \cdot z_2| = |z_1| \cdot |z_2|$$
$$\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$$
$$|z^m| = |z|^m$$

Properties of Arguments:

$$\mathrm{Arg}(z_1 \cdot z_2) = \mathrm{Arg}(z_1) + \mathrm{Arg}(z_2)$$
$$\mathrm{Arg}\left(\frac{z_1}{z_2}\right) = \mathrm{Arg}(z_1) - \mathrm{Arg}(z_2)$$
$$\mathrm{Arg}(z^m) = m\,\mathrm{Arg}(z)$$

`https://scipp.ucsc.edu/~haber/ph116A/arg_11.pdf`

Express each of the following complex numbers in polar form:

(a) $2 + 2\sqrt{3}i$

(b) $-5 + 5i$

Problem

Solve the equation: $e^{4z} = i$

Problem

Solve for x *and* y,

$$\left(\frac{3}{2} + \frac{\sqrt{3}}{2}i\right)^{2024} = 3^{1012}(x + iy)$$

Good-bye

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