MAT215: Complex Variables And Laplace Transformations

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LECTURE-04

Motivation: Real vs. Complex Roots

In $\mathbb{R}$:

- For $a > 0$, the equation $x^n = a$ has:
  - **one** positive real root if n is even,
  - **one** real root if n is odd.
- Roots can be chosen *continuously* on $(0, \infty)$: $x = \sqrt[n]{a}$.
- No ambiguity from angles; order is total; $\mathbb{R}$ is simply connected.

In $\mathbb{C}$:

- For $w \neq 0$, the equation $z^n = w$ has **exactly** n distinct roots.
- Formula uses *multi-valued argument*: $\theta \sim \theta + 2\pi k$.
- Any continuous branch must *exclude a ray* (branch cut) from 0.
- Going once around 0 $\Rightarrow$ you land on a *different branch*.

Polar Form & the Source of Multivaluedness

Every nonzero $w \in \mathbb{C}$ can be written as

$$w = r\,e^{i\theta}, \qquad r = |w| > 0, \quad \theta = \arg w.$$

But arg is **not single-valued**:

$$\arg w = \theta + 2\pi k, \quad k \in \mathbb{Z}.$$

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$$\Rightarrow \quad w^{1/n} = r^{1/n}\,e^{i(\theta+2\pi k)/n}, \quad k = 0, 1, \ldots, n-1.$$

- All n roots lie on the circle of radius $r^{1/n}$, equally spaced at angle $2\pi/n$.
- The "strangeness": arg forces n consistent choices (*branches*) to define $z \mapsto z^{1/n}$.

Geometry: Roots as a Regular Polygon

Takeaway

For each fixed $w \neq 0$, the n-th roots form a regular n-gon centered at the origin with radius $|w|^{1/n}$.

Monodromy of $z^{1/3}$: an explicit loop example

Let $f(z) = z^{1/3}$ with the principal argument $\operatorname{Arg} z \in (-\pi, \pi]$ (branch cut along $(-\infty, 0]$). Write $z = re^{i\theta} \Rightarrow f(z) = r^{1/3} e^{i\theta/3}$.

Start. Take $z_0 = 1 = e^{i\cdot 0}$. Then $f(z_0) = 1^{1/3} e^{i\cdot 0} = 1$.

One full loop around the origin. Move along the unit circle $z = e^{i\theta}$ as $\theta : 0 \to 2\pi$.

$$f_{\text{after 1 loop}} = e^{i(\theta+2\pi)/3} = e^{i\theta/3} \quad \underbrace{e^{i2\pi/3}}_{\text{rotation by } 120^\circ}$$

Thus f is multiplied by $e^{i2\pi/3} = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$: we land on a <u>different branch</u>.

Two and three loops.

$$\text{after 2 loops: } f \mapsto f\, e^{i4\pi/3}, \qquad \text{after 3 loops: } f \mapsto f\, e^{i6\pi/3} = f\, e^{i2\pi} = f$$

Only after $\boxed{\text{3 loops}}$ do we return to the original value.
`https://www.geogebra.org/m/rrnT76DX`.

Takeaway (monodromy)

Each circuit adds 2π to the argument, so $z^{1/3}$ is multiplied by $e^{2\pi i/3}$. The three branches form a 3-sheeted helical Riemann surface around 0. (For $z^{1/n}$: rotation $e^{2\pi i/n}$; return after n loops.)