MAT215: Complex Variables And Laplace Transformations

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Lecture-06

Motivation: Limit in Higher Dimensions

- In single-variable calculus, we know

$$\lim_{x \to a} f(x) = L \iff f(x) \text{ can be made as close as we want to } L \text{ when } x \text{ is}$$

- But what happens when x is replaced by a vector (x, y) or a complex number z?
- The challenge: there are **infinitely many paths** approaching a point.

Example 1: Different Paths, Different Limits

Consider

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We want to know whether $\lim_{(x,y)\to(0,0)} f(x,y)$ exists.

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- Along $y = 0$: $f(x,0) = 0$
- Along $y = x^2$: $f(x,x^2) = \frac{x^4}{2x^4} = \frac{1}{2}$

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$\Rightarrow$ Limit does not exist since it depends on the path.

Example 2: Polar Coordinate Technique

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As $r \to 0$, $f(r,\theta) \to 0$ regardless of θ.

$$\boxed{\text{Hence, } \lim_{(x,y)\to(0,0)} f(x,y) = 0.}$$

Example 3: Oscillation near the Origin

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As $r \to 0$, $\frac{\sin\left(r^2\right)}{r^2} \to 1$.

$$\boxed{\text{Limit exists and equals 1.}}$$

Limits in $\mathbb{C}$

- A function $f : \mathbb{C} \to \mathbb{C}$ has a limit L at z_0 if:

$$\forall \varepsilon > 0, \exists \delta > 0 \text{ such that } |f(z) - L| < \varepsilon \text{ whenever } 0 < |z - z_0| < \delta.$$

- The same issue arises: z can approach z_0 from any direction in the plane.
- So path independence is essential.

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Observation: Polynomials are continuous in $\mathbb{C}$ just like in $\mathbb{R}$.

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Approaching along real axis $y = 0$: $f(x,0) = 1$.
Approaching along imaginary axis $x = 0$: $f(0,y) = \frac{y^2}{-y^2} = -1$.

Hence, limit does not exist.

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Observation: Although $f(z) = |z|^2$ is not complex differentiable, the limit exists.

Example 4

If

$$f(z) = \begin{cases} \frac{z^2-4}{z^2-3z+2}, & z \neq 2 \\ kz^2+6, & z = 2 \end{cases}$$

Find k such that such that $f(z)$ is continuous at $z = 2$.

Summary and Reflection

- In $\mathbb{R}^2$ and $\mathbb{C}$, limit existence requires the same value along every path.
- Polar coordinates help detect limit behavior efficiently.
- In $\mathbb{C}$, algebraic expressions behave like $\mathbb{R}^2$ functions—but differentiability is far stricter.
- Next Topic: **Continuity and differentiability in** $\mathbb{C}$.

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$$\operatorname{Re}(z) = r\cos\theta, \quad \operatorname{Im}(z) = r\sin\theta, \quad |z| = r.$$

$$f(z) = \frac{r^2\cos\theta\sin\theta}{r^2} = \cos\theta\sin\theta.$$

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Hence: Limit as $z \to 0$ depends on the angle of approach θ.

$$\boxed{\text{No limit exists.}}$$

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Note: This function has no limit (and no derivative) at 0, but its magnitude $|f(z)| = 1$ is constant.

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$$|f(z)| \to 0 \quad \text{as} \quad z \to 0.$$

Therefore,

$$\boxed{\lim_{z \to 0} f(z) = 0.}$$

Even though limit exists, f is not differentiable at 0 because of the $|z|^2$ term (depends on both z and $\overline{z}$).

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But f is <u>not</u> continuous at $z = 0$ if extended by $f(0) = 0$? Actually it is continuous—but still not differentiable at 0.