L'Hospital's Rule and Indeterminate Forms

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1. Motivation

When evaluating limits, sometimes substitution yields undefined forms such as

$$\frac{0}{0}, \quad \frac{\infty}{\infty}, \quad 0 \cdot \infty, \quad \infty - \infty, \quad 0^0, \quad 1^\infty, \quad \infty^0.$$

These are called **indeterminate forms** since the resulting limit depends on the relative rates of vanishing or growth of the functions involved.

Theorem

Let f, g be differentiable near a (except possibly at a). Suppose:

1. $f(a) = g(a) = 0$ or $f(x), g(x) \to \infty$ as $x \to a$,
2. $g'(x) \neq 0$ near a,
3. $\lim_{x \to a} \frac{f'(x)}{g'(x)}$ exists (finite or ∞).

Then

$$\boxed{\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}}.$$

2. Indeterminate Form $\frac{0}{0}$

Example

$$\lim_{x \to 0} \frac{\sin x}{x} \quad \Rightarrow \quad \frac{0}{0}.$$

Apply L'Hospital:

$$\lim_{x \to 0} \frac{\cos x}{1} = 1.$$

Example

$$\lim_{x \to 0} \frac{e^x - 1}{x} = \lim_{x \to 0} \frac{e^x}{1} = 1.$$

3. Indeterminate Form $\frac{\infty}{\infty}$

Example

$$\lim_{x\to\infty} \frac{3x^2+4x}{5x^2-7} \Rightarrow \frac{\infty}{\infty}.$$

Differentiate:

$$\lim_{x\to\infty} \frac{6x+4}{10x} = \frac{3}{5}.$$

Example

$$\lim_{x\to\infty} \frac{\ln x}{x} = \lim_{x\to\infty} \frac{1/x}{1} = 0.$$

4. Indeterminate Form $0 \cdot \infty$

Example

$$\lim_{x\to 0^+} x\ln x = \lim_{x\to 0^+} \frac{\ln x}{1/x}.$$

Apply L'Hospital:

$$\lim_{x\to 0^+} \frac{1/x}{-1/x^2} = \lim_{x\to 0^+} (-x) = 0.$$

5. Indeterminate Form $\infty - \infty$

Example

$$\lim_{x\to\infty} (\sqrt{x^2+x} - x)$$

Multiply by conjugate:

$$\frac{x}{\sqrt{x^2+x}+x} = \frac{1}{\sqrt{1+1/x}+1} \to \frac{1}{2}.$$

6. Exponential Indeterminate Forms

Example

$$\lim_{x\to 0^+} x^x.$$

Let $\ln y = x\ln x$. Since $x\ln x \to 0$,

$$\ln y \to 0 \Rightarrow y \to e^0 = 1.$$

Example

$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x.$$

Let $\ln y = x\ln(1+1/x)$.

$$\lim_{x\to\infty} x\left(\frac{1}{x}-\frac{1}{2x^2}+\cdots\right)=1,$$

so $y=e$.

Example

$$\lim_{x\to 0^+}\left(\frac{1}{x}\right)^x.$$

$\ln y = -x\ln x \to 0 \Rightarrow y \to 1$.

7. Repeated Applications

Example

$$\lim_{x\to 0}\frac{1-\cos x}{x^2}.$$

After two applications of L'Hospital:

$$\lim_{x\to 0}\frac{\cos x}{2}=\frac{1}{2}.$$

Example

$$\lim_{x\to 0}\frac{e^{2x}-1-2x}{x^2}=\lim_{x\to 0}\frac{2e^{2x}}{1}=2.$$

8. Summary Table

Type	**Example**	**Technique**	**Result**
$0/0$	$\frac{\sin x}{x}$	Direct L'Hospital	1
∞/∞	$\frac{\ln x}{x}$	Direct L'Hospital	0
$0\cdot\infty$	$x\ln x$	Rewrite as $\ln x/(1/x)$	0
$\infty-\infty$	$\sqrt{x^2+x}-x$	Multiply by conjugate	$1/2$
0^0	x^x	Logarithmic form	1
1^∞	$(1+1/x)^x$	Logarithmic form	e
∞^0	$(1/x)^x$	Logarithmic form	1

Note

Key Ideas:

- L'Hospital's Rule compares *rates of change* of numerator and denominator.
- Sometimes algebraic simplification is easier than differentiation.
- Always ensure $g'(x) \neq 0$ and both functions are differentiable in a neighborhood of a.

Note

When both f and g vanish (or blow up), their ratio is governed by how steeply each function changes near the point:

$$\frac{f(x)}{g(x)} \approx \frac{f'(x)}{g'(x)}.$$