MAT215: Complex Variables And Laplace Transformations

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LECTURE-08

Differentiability in higher dimensions

What does it mean by a function, $f : D \subset \mathbb{R} \to \mathbb{R}$, is differentiable at a point a?

Geometrically, what does it mean?

$$\boxed{\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}} \qquad \boxed{\lim_{x \to a} \frac{f(x) - f(a)}{x - a}}$$

Question: How can you lift this definition to a higher dimension?

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Question: How can you lift this definition to a higher dimension?

Problem-1: Dividing by the norm of $(x - a) \in \mathbb{R}^n$

Problem-2: Considering the derivative as a transformation: $\tilde{f}'(a) \cdot (x - a)$

Definition

Definition

The function $f : \mathbb{R}^n \to \mathbb{R}^m$ is differentiable at the point a if there exists a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ that satisfies the condition:

$$\boxed{\lim_{x \to a} \frac{\|f(x) - f(a) - T(x-a)\|}{\|x-a\|} = 0}$$

You can check https://mathinsight.org/differentiability_multivariable_definition to get the full insight about the condition.

Find the Derivative

Example

Let $f : \mathbb{R}^2 \to \mathbb{R}$, and suppose the function is differentiable at $a = (x_0, y_0)$ and the derivative (transformation) is denoted by $Df(x_0, y_0)$. Then the condition gives us:

$$\lim_{(x,y)\to(x_0,y_0)} \frac{\|f(x,y) - f(x_0,y_0) - Df(x_0,y_0)(x - x_0, y - y_0)\|}{\|x - a\|} = 0$$

Now, we knew that $Df(x_0, y_0)$ must be something like $\begin{pmatrix} f_1(x_0,y_0) & f_2(x_0,y_0) \end{pmatrix}$. Consider, $y = y_0$ then,

$$\lim_{x\to x_0} \frac{\|f(x,y_0) - f(x_0,y_0) - f_1(x_0,y_0)(x - x_0) - f_2(x_0,y_0)(y_0 - y_0)\|}{\sqrt{(x - x_0)^2 + (y_0 - y_0)^2}} = 0$$

$$\lim_{x\to x_0} \frac{\|f(x,y_0) - f(x_0,y_0) - f_1(x_0,y_0)(x - x_0)\|}{\|x - x_0\|} = 0$$

Continued...

Example

$$\underbrace{\lim_{x\to x_0} \frac{\|f(x,\star)-f(x_0,\star)-f_1(x_0,\star)(x-x_0)\|}{\|x-x_0\|}=0}_{f_1(x_0,y_0)=\partial_1 f(x_0,y_0)}$$

Similarly, we can get,

$$\underbrace{\lim_{y\to y_0} \frac{\|f(\star,y)-f(\star,y_0)-f_2(\star,y_0)(y-y_0)\|}{\|y-y_0\|}=0}_{f_2(x_0,y_0)=\partial_2 f(x_0,y_0)}$$

Hence,

$$Df(x_0,y_0)=\begin{pmatrix}\partial_1 f(x_0,y_0) & \partial_2 f(x_0,y_0)\end{pmatrix}$$

Which is nothing but the Jacobian.

Complex Derivative

Question: What is $f'(a)$ in the definition of the complex derivative? Consider the complex function as $f : \mathbb{R}^2 \to \mathbb{R}^2$ then we can mimic the same computation and will get:

$$Df = \begin{pmatrix} \partial_1 u & \partial_2 u \\ \partial_1 v & \partial_2 v \end{pmatrix}$$

which should follow something like, $\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$. Which implies,

$$\begin{cases} \partial_x u & = \partial_y v \\ -\partial_y u & = \partial_x v \end{cases}$$

This is nothing but the famous Cauchy-Riemann Equation.

CR Theorem

Definition

A complex function $f(z) = u(x,y) + iv(x,y)$ has a complex derivative $f'(z)$ if and only if its real and imaginary part are continuously differentiable and satisfy the Cauchy-Riemann equations

$$u_x = v_y, \quad u_y = -v_x$$

In this case, the complex derivative of $f(z)$ is equal to any of the following expressions:

$$f'(z) = \underbrace{u_x + iv_x}_{\text{horizontal dir.}} = \underbrace{v_y - iu_y}_{\text{vertical dir.}}$$

Once CR equations hold, any direction of approach yields the same derivative. Check `https://complex-analysis.com/content/complex_differentiation.html` for more details.

Holomorphic vs Analytic

Holomorphic is just talking about differentiability in the complex plane. Suppose a complex-valued function is holomorphic in all points of a domain, then it is called an analytic function. Have a look at the Wiki `https://en.wikipedia.org/wiki/Holomorphic_function`.

Wirtinger Derivative

The Wirtinger derivatives are defined as the following linear partial differential operators of first order:

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)$$
$$\frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$$

Examples

Example

Consider the function $f : \mathbb{C} \to \mathbb{C}$, $f(z) = z^2$. Find the complex derivative.

The $f(z)$ can be written as

$$z^2 = \left(x^2 - y^2\right) + i(2xy)$$

Its real part $u = x^2 - y^2$ and imaginary part $v = 2xy$ satisfy the Cauchy-Riemann equations, since

$$u_x = 2x = v_y, \quad u_y = -2y = -v_x$$

The CR theorem implies that $f(z) = z^2$ is differentiable. Its derivative turns out to be

$$f'(z) = u_x + iv_x = v_y - iu_y = 2x + i2y = 2(x + iy) = 2z$$