Linear Algebra & Fourier Analysis

Assignment 01

Total - 50 Marks, Due date: March 8

(You need to answer all questions except bonus)

---

1. (a) Solve the following system,

$$\begin{cases} x + 4z + 7t &= 0 \\ y + 5z + 11t &= 0 \end{cases}$$

If the right hand side is zero, we call it a homogeneous system like above.

(b) Now, if we change the right hand side as,

$$\begin{cases} x + 4z + 7t &= 8 \\ y + 5z + 11t &= 11 \end{cases}$$

Can you find the general solution?
Hint: You don't need to solve it explicitly. Think wisely.

(c) So, we solve two systems:

$$\left[\begin{array}{cccc|c} 1 & 0 & 4 & 7 & 0 \\ 0 & 1 & 5 & 11 & 0 \end{array}\right] \quad \text{and} \quad \left[\begin{array}{cccc|c} 1 & 0 & 4 & 7 & 8 \\ 0 & 1 & 5 & 11 & 11 \end{array}\right]$$

We can consider them as, $A\vec{x} = \vec{0}$ and $A\vec{x} = \vec{b}$ where $\vec{b} = \begin{pmatrix} 8 \\ 11 \end{pmatrix}$. How can you describe the relation between the solution set of $A\vec{x} = \vec{b}$ and the solution set of $A\vec{x} = \vec{0}$.

(5 Marks)

2. Find conditions on α, β, γ for the following system to be consistent:

$$\begin{aligned} x_1 + 2x_2 - x_3 &= \alpha \\ 2x_1 + 5x_2 + 3x_3 &= \beta \\ -x_1 - 4x_2 - 2x_3 &= \gamma \end{aligned}$$

   Furthermore, describe how the system behaves when the consistency condition is not satisfied.

   Hint: Convert the augmented matrix to row echelon form and analyze the last row.

   (5 Marks)

3. Find the reduced row echelon form of the augmented matrix for the following system:

$$\begin{aligned} 3x + 5y - 2z + 7t &= 4 \\ -6x + y + 3z - 2t &= -1 \\ 9x - 4y - 5z + 8t &= 7 \end{aligned}$$

   Identify the pivot columns and determine whether any variables are free. If so, express the solution set in parametric form. (5 Marks)

4. Take your student **ID** as a number. Let **ID4** represent the 4th digit of your **ID**. For example, if 87654321 is your **ID** then **ID2** is 7.

   (a) Now write the corresponding system of equation of the following augmented matrix:

$$\left(\begin{array}{cccc|c} \mathbf{ID8} & 0 & 0 & \mathbf{ID8} & \mathbf{ID8} \\ 0 & \mathbf{ID7} & \mathbf{ID6} & 0 & \mathbf{ID7} \\ \mathbf{ID3} & \mathbf{ID7} & \mathbf{ID6} & \mathbf{ID3} & \mathbf{ID3} + \mathbf{ID7} \end{array}\right)$$

   (b) Find the parametric solution as well as the general solution (if possible) of that system.

   (c) Find two random solution $\mathbf{u}$ and $\mathbf{v}$ and show that if $\mathbf{u} + \mathbf{v}$ is also a solution of the system or not?

   (2+7+1 Marks)

5. Find the inverse of the following matrix using Gaussian Elimination:

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 5 & 6 \\ 2 & 5 & 7 \end{pmatrix}$$

   Wait, let's have some fun. I know you apply the four row operation to find the inverse. Now, apply each of those four row operation to the identity matrix I_3 and label them as E_1, E_2, E_3 and E_4. And verify that, $E_4E_3E_2E_1A = I_3$. Can you describe why it happen? And how can you write the inverse matrix A^{-1} in terms of E_1, E_2, E_3 and E_4 matrices. (15 Marks)

6. (a) Find the values of k when the following systems will be consistent:

$$\left(\begin{array}{cc|c} 3 & -4 & k \\ -6 & 8 & 5 \end{array}\right), \left(\begin{array}{cc|c} k & 1 & -2 \\ 4 & -1 & 2 \end{array}\right)$$

(b) Find the value of λ and μ such that the following system of equations have (i) No solution, (ii) Many solution, and (iii) Unique solution (if possible):

$$\begin{cases} x + y + z & = 6 \\ x + 2y + 3z & = 10 \\ x + 2y + \lambda z & = \mu \end{cases}, \qquad \begin{cases} x + y + z & = 6 \\ y + 2z & = 4 \\ (\lambda - 3)z & = \mu - 10 \end{cases}$$

(10 Marks)

Bonus Question:

1. (a) Let t be an arbitrary real number and let,

$$s_1 = -\frac{3}{2} - 2t$$
$$s_2 = \frac{3}{2} + t$$
$$s_3 = t$$

Is the list (s_1, s_2, s_3) a solution to the linear system for any choice of parameter t? If not, then for which value of t will it become?

$$x_1 + x_2 + x_3 = 0$$
$$x_1 + 3x_2 - x_3 = 0$$

(b) Suppose we have given the null space (remember I introduced it as nullifying vectors which appear inside our general solution) of a matrix which was in the reduced row echelon form:

$$N = \alpha \begin{pmatrix} 7 \\ 0 \\ 0 \\ 8 \\ 0 \\ 0 \\ 3 \\ -1 \end{pmatrix} + \beta \begin{pmatrix} 3 \\ 0 \\ 0 \\ 4 \\ 0 \\ -1 \\ 0 \\ 0 \end{pmatrix} + \gamma \begin{pmatrix} 2 \\ 0 \\ 0 \\ 3 \\ -1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \delta \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Now, can you predict that linear system?
Hint: Can you identify the pivot columns? Look carefully at the nullifying vectors and what information they contain (like how they interact with the pivots).

(2+3 Marks)

Best of Luck!