BRAC UNIVERSITY

Inspiring Excellence

Linear Algebra & Fourier Analysis

Assignment 03

Total - 90 Marks, Due date: Sunday, May 11

(You need to answer three questions from each section)

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1 Inner Product

1. Let $\langle \mathbf{u}, \mathbf{v} \rangle = \mathbf{u}^T A \mathbf{v}$ where

$$A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}, \quad \mathbf{u}, \mathbf{v} \in \mathbb{R}^2.$$

   Check whether this defines a valid inner product on $\mathbb{R}^2$. (10 Marks)

2. Let $f(x) = x$, $g(x) = x^2$ defined on $[0, 1]$. Compute their inner product in the space $C[0, 1]$ with the inner product

$$\langle f, g \rangle = \int_0^1 f(x)g(x)\, dx.$$

   Show that $\sin(x)$ and $\cos(x)$ are orthogonal in this inner product space. (10 Marks)

3. Let $A = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix}$. Define inner product on $M_{2\times 2}(\mathbb{R})$ by

$$\langle A, B \rangle = \text{trace}(A^T B).$$

   Compute $\langle A, B \rangle$. (10 Marks)

4. Apply the Gram-Schmidt process to the functions $f_1(x) = 1$, $f_2(x) = x$, $f_3(x) = x^2$ over $[0, 1]$ with inner product $\langle f, g \rangle = \int_0^1 f(x)g(x)\, dx$, and compute the orthonormal set up to f_2. (10 Marks)

5. Apply Gram-Schmidt to orthogonalize the set $\{(1,1,1,1),(1,1,-1,-1),(1,-1,-1,1)\}$ in $\mathbb{R}^4$. Normalize the result to get an orthonormal basis. (10 Marks)

6. Find the orthogonal complement $W^\perp$ using the standard inner product of the subspace,

$$W = \text{Span}\left\{ \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 4 \end{pmatrix} \right\}$$

Hint: $\forall u \in W, \forall v \in W^\perp$ we have $\langle u, v \rangle = 0$. Okay, then consider $\begin{pmatrix} x \\ y \\ z \end{pmatrix} \in W^\perp$, now calculate the inner product and find the subspace. (10 Marks)

2 Fourier series

1. Consider the function,

$$f(x) = \begin{cases} -x, -\pi < x < 0 \\ x, 0 < x < \pi \end{cases}$$

   (a) Determine whether the function is even, odd or neither even nor odd.

   (b) Find the Fourier series expansion of the function $f(x)$.

(10 Marks)

2. Expand $f(x) = \cos(x), 0 < x < \pi$ in a Fourier sine series. (10 Marks)

3. (a) Expand $f(x) = x, 0 < x < 2$ in a half range series of cosines.

   (b) Using the parseval's identity for this case and hence deduce the value of $S = \sum_{n=1}^{\infty} \frac{1}{n^4}$.

(10 Marks)

4. Expand $f(x) = x^2, 0 < x < 2\pi$, in a Fourier series if the period is 2π. (10 Marks)

5. Expand

$$f(x) = \begin{cases} x, & 0 < x < 4 \\ 8 - x, & 4 < x < 8 \end{cases}$$

in a Fourier series of cosines. (10 Marks)

6. Use euler identity to prove that the complex form of Fourier series can be expressed as

$$\sum_{n=-\infty}^{\infty} c_n e^{\frac{n\pi x i}{L}}$$

(10 Marks)

3 Fourier Transform

1. Find the Fourier cosine transform of $e^{-mx}, x \geq 0$. Hence show that,

$$\int_0^\infty \frac{\beta \cos(pv)}{v^2 + \beta^2} dv = \frac{\pi}{2} e^{-p\beta}, p > 0, \beta > 0$$

(10 Marks)

2. Find the Fourier transform of the function,

$$f(x) = \begin{cases} 1 - x^2, |x| \leq 1 \\ 0, |x| > 1 \end{cases}$$

Hence, determine

$$\int_0^\infty \frac{\sin^4 x}{x^2} dx = \frac{\pi}{4}$$

(10 Marks)

3. Find the Fourier transform of the function,

$$f(x) = \begin{cases} 1, |x| < 1 \\ 0, |x| > 1 \end{cases}$$

Hence, show that

$$\int_0^\infty \frac{\sin x}{x} dx = \frac{\pi}{2}$$

(10 Marks)

4. Find the Fourier transform of the function,

$$f(x) = \begin{cases} 1, |x| < a \\ 0, |x| > a \end{cases}$$

Hence, show that

$$\int_0^\infty \frac{\sin(ax)\cos(ax)}{x} dx = \frac{\pi}{4}$$

(10 Marks)

5. Find the Fourier transform of the function,

$$f(x) = \begin{cases} 1 - x^2, |x| < 1 \\ 0, |x| > 1 \end{cases}$$

Hence, evaluate the definite integral

$$\int_0^\infty \frac{x\cos(x) - \sin(x)}{x^3} \cos\left(\frac{x}{2}\right) dx$$

(10 Marks)

6. Find the Fourier transform of the function,

$$f(x) = \begin{cases} 1 - |x|, |x| < 1 \\ 0, |x| > 1 \end{cases}$$

Hence, evaluate the definite integral

$$\int_0^\infty \frac{\sin^2 x}{x^2} dx$$

(10 Marks)

Best of Luck!