BRAC UNIVERSITY
Inspiring Excellence

Linear Algebra & Fourier Analysis

Final Practice sheet

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1 Inner Product

1. Find the eigenspace of the following matrix, draw the eigenspace and diagonalize the matrix

$$\begin{pmatrix} 6 & -1 \\ 2 & 3 \end{pmatrix}$$

(10 Marks)

Answer: First, get the eigenvalues and eigenvectors,

$$\text{For } \lambda = 5 : \vec{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\text{For } \lambda = 4 : \vec{v}_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

Okay, then eigenspace is the space generated by the corresponding eigenvectors. Like for $\lambda = 5$, it's the line $y = x$ because $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ vector generate this line. Similarly, do for $\lambda = 4$.

2. Let $A = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix}$. Define inner product on $M_{2\times 2}(\mathbb{R})$ by

$$\langle A, B \rangle = \text{trace}(A^T B).$$

Compute $\langle A, B \rangle$. (10 Marks)

Answer: Do the computation.

3. (a) Illustrate how the inner product is related to the orthogonal projection.

(b) Apply the Gram-Schmidt process to the functions $f_1(x) = 1$, $f_2(x) = x$, $f_3(x) = x^2$ over $[0, 1]$ with inner product $\langle f, g\rangle = \int_0^1 f(x)g(x)\,dx$, and compute the orthonormal set up to f_2.

(10 Marks)

Answer:

(a) Let V be an inner product space, and let $u \in V$ be a vector (or function), and $v \in V$ be a nonzero vector (or function) onto which we want to project u.

The orthogonal projection of u onto v is given by:

$$\mathrm{proj}_v(u) = \frac{\langle u, v\rangle}{\langle v, v\rangle} v$$

When we subtract this projected part from u, what we left is the orthogonal part with respect to v. Hence, $u - \mathrm{proj}_v(u)$ is orthogonal to v or

$$\langle u - \mathrm{proj}_v(u), v\rangle = 0$$

(b) Do the computation.

4. Apply Gram-Schmidt to orthogonalize the set $\{(1, 1, 1, 1), (1, 1, -1, -1), (1, -1, -1, 1)\}$ in $\mathbb{R}^4$. Normalize the result to get an orthonormal basis. (10 Marks)

5. Show that $f(x) = \sin(x)$ and $g(x) = \cos(x)$ are orthogonal in the interval $[-\pi, \pi]$.(10 Marks)

   Answer: Show that,

$$\frac{1}{\pi}\int_{-\pi}^{\pi} sin(x)\cos(x)dx = 0$$

6. Find the orthogonal complement $W^\perp$ using the standard inner product of the subspace,

$$W = \mathrm{Span}\left\{ \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 4 \end{pmatrix} \right\}$$

   Hint: $\forall u \in W, \forall v \in W^\perp$ we have $\langle u, v\rangle = 0$. Okay, then consider $\begin{pmatrix} x \\ y \\ z \end{pmatrix} \in W^\perp$, now calculate the inner product and find the subspace. (10 Marks)

2 Fourier series

1. Consider the function,

$$f(x) = \begin{cases} x, -\pi < x < 0 \\ -x, 0 < x < \pi \end{cases}$$

(a) Determine whether the function is even, odd or neither even nor odd.

(b) Find the Fourier series expansion of the function $f(x)$.

Answer:

(a) Even

(b)

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos(kx)$$

$$a_k = \frac{2}{\pi}\int_0^{\infty} x\cos(kx)dx$$

(10 Marks)

2. Expand $f(x) = \cos(x), 0 < x < \pi$ in a Fourier sine series. (10 Marks)

3. (a) Expand $f(x) = x, 0 < x < 2$ in a half range series of cosines.

   (b) Using the parseval's identity for this case and hence deduce the value of $S = \sum_{n=1}^{\infty} \frac{1}{n^4}$.

Answer:

First of all, for the cosine series, we have $b_k = 0$.

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos\left(\frac{k\pi x}{2}\right)$$

$$a_k = \frac{2}{2}\int_0^2 x\cos\left(\frac{k\pi x}{2}\right)dx = \frac{4}{k^2\pi^2}\left((-1)^k - 1\right)$$

Using the inner product on the interval $[-L, L] = [-2, 2]$,

$$\langle f, g\rangle = \frac{1}{L}\int_{-L}^{L} f(x)g(x)\,dx$$

we apply Parseval's identity:

$$\frac{1}{2}\int_{-2}^{2} x^2\,dx = \frac{a_0^2}{2} + \sum_{k=1}^{\infty} a_k^2$$

Fill up the computation. (10 Marks)

4. Expand $f(x) = x^2, 0 < x < 2\pi$, in a Fourier series if the period is 2π. (10 Marks)

Answer: $2L = 2\pi \implies L = \pi$

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos(kx) + \sum_{k=1}^{\infty} b_k \sin(kx)$$

$$a_k = \frac{1}{\pi}\int_0^{2\pi} x^2 \cos{(kx)}dx$$
$$b_k = \frac{1}{\pi}\int_0^{2\pi} x^2 \sin{(kx)}dx$$

5. Expand
$$f(x) = \begin{cases} x, & 0 < x < 4 \\ 8 - x, & 4 < x < 8 \end{cases}$$
in a Fourier series of cosines. (10 Marks)

Answer: Here, $L = 8$ because it's the half interval math. Actually the full interval was $[-L, L] = [-8, 8]$
$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos\frac{k\pi x}{8}$$
$$a_k = \frac{2}{8}\int_0^8 f(x)\cos{(kx)}dx = \frac{2}{8}\left[\int_0^4 x\cos\frac{k\pi x}{8}dx + \int_4^8 (8-x)\cos\frac{k\pi x}{8}dx\right]$$

6. Use euler identity to prove that the complex form of Fourier series can be expressed as
$$\sum_{n=-\infty}^{\infty} c_n e^{\frac{n\pi x i}{L}}$$
(10 Marks)

7. Find the complex form of the Fourier series of $f(x) = x$ on the interval $[-2, 2]$. (10 Marks)

Answer:
$$f(x) = \sum_{-\infty}^{\infty} c_n e^{\frac{n\pi x i}{2}}$$
Where,
$$C_n = \frac{1}{2L}\int_{-L}^{L} f(x)e^{\frac{-in\pi x}{L}}dx$$
$$= \frac{1}{2\cdot 2}\int_{-2}^{2} xe^{\frac{-in\pi x}{2}}dx$$

3 Fourier Transform

1. Find the Fourier cosine transform of $e^{-mx}, x \geq 0$. Hence show that,
$$\int_0^{\infty} \frac{\beta\cos(pv)}{v^2+\beta^2}dv = \frac{\pi}{2}e^{-p\beta}, p > 0, \beta > 0$$

(10 Marks)

Answer: Check here https://drive.google.com/file/d/1dZqfmH_ybkuMVyoWLAqcTKv9-YPeRi15/view

2. Find the Fourier transform of the function,

$$f(x) = \begin{cases} 1 - x^2, |x| \leq 1 \\ 0, |x| > 1 \end{cases}$$

Hence, determine

$$\int_0^\infty \frac{\sin^4 x}{x^2} dx = \frac{\pi}{4}$$

(10 Marks)

Answer: This question is wrong. The actual integral will be, determine

$$\int_0^\infty \left(\frac{x\cos(x) - \sin(x)}{x^3} \cos\left(\frac{x}{2}\right)\right) dx$$

Check here for the solution, https://drive.google.com/file/d/1vb-YkKBkbjmkt-ypRnNPALLnQDGei view

3. Find the Fourier transform of the function,

$$f(x) = \begin{cases} \pi, |x| < 1 \\ 0, |x| > 1 \end{cases}$$

Hence, show that

$$\int_0^\infty \frac{\sin x}{x} dx = \frac{\pi}{2}$$

(10 Marks)

Answer: Check here for the solution, https://drive.google.com/file/d/1vb-YkKBkbjmkt-ypRnNP view

4. Find the Fourier transform of the function,

$$f(x) = \begin{cases} 1, |x| < a \\ 0, |x| > a \end{cases}$$

Hence, show that

$$\int_0^\infty \frac{\sin(ax)\cos(ax)}{x} dx = \frac{\pi}{4}$$

(10 Marks)

5. Find the Fourier transform of the function,

$$f(x) = \begin{cases} 1 - x^2, |x| < 1 \\ 0, |x| > 1 \end{cases}$$

Hence, evaluate the definite integral

$$\int_0^\infty \frac{x\cos(x) - \sin(x)}{x^3} \cos\left(\frac{x}{2}\right) dx$$

(10 Marks)

Answer: Duplicate question 2.

6. Find the Fourier transform of the function,

$$f(x) = \begin{cases} 1 - |x|, |x| < 1 \\ 0, |x| > 1 \end{cases}$$

Hence, evaluate the definite integral

$$\int_0^\infty \frac{\sin^2 x}{x^2} dx$$

(10 Marks)

Answer: Check here for the solution, `https://drive.google.com/file/d/1vb-YkKBkbjmkt-ypRnNP` `view`

Best of Luck!