Practice problem

1. Find the solution of the following system of linear system using **Gauss Jordan Elimination**,

$$\begin{cases} x + 3y + z + t &= 2 \\ 2x + 6y + 3z + 4t &= 5 \\ 7x + 21y + 8z + 9t &= 15 \end{cases}$$

Answer:

$$\begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} = \begin{pmatrix} 1 - 3y - t \\ y \\ 1 - 2t \\ t \end{pmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + y \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 0 \\ -2 \\ 1 \end{bmatrix}$$

2. Calculate the inverse of the following matrix using Gauss-Jordan elimination (i.e. using reduced row echelon form),

$$A = \begin{pmatrix} 1 & 0 & -1 \\ 2 & 1 & -2 \\ 0 & 2 & 1 \end{pmatrix}$$

3. Is the set of vectors $W = \left\{ \begin{bmatrix} a \\ b \\ c \end{bmatrix} \in \mathbb{R}^3 \middle| a + b + c = 0 \right\}$ a subspace of the vector space $V = \mathbb{R}^3$.

Answer: Yes. Check $u + v \in W$ and $\alpha u \in W$.

4. Let V be the set of all ordered pairs of real numbers and consider the following addition and scalar multiplication operations defined on the ordered pairs, $x = (x_1, x_2)$ and $y = (y_1, y_2)$:

$$x + y = (x_1 + x_2 + 1, y_1 + y_2 + 1), \quad \alpha \cdot x = (\alpha x_1, \alpha x_2)$$

Answer: No. because $\alpha(u + v) \neq \alpha u + \alpha v$ where $u = (u_1, u_2), v = (v_1, v_2)$ and $\alpha \in \mathbb{R}$.

5. Consider the vector space V of 2 by 2 matrix, $M_{2\times2}$. Now, Consider the following subsets:

    1. $W_1 = \{M_{2\times2} : \det M_{2\times2} = 0\}$
    2. $W_2 = \{M_{2\times2} : \det M_{2\times2} \neq 0\}$
    3. $W_3 = \{M_{2\times2} : \det M_{2\times2} = 1\}$

Verify which of the subsets are subspace of V. Explain your answer.

Answer: None of them are subspace. Try to construct a counterexample from the following axioms: $\bar{0} + M = M$ and $1 \cdot M = M$.

6. Check whether the set of solutions of the homogeneous linear system is a subspace of the vector space where A is an $n \times n$ matrix.
Answer: Given an $n \times n$ matrix A, consider the homogeneous linear system:

$$A\mathbf{x} = \mathbf{0}$$

where $\mathbf{x}$ is an n-dimensional vector. The set of solutions to this system is:

$$V = \{\mathbf{x} \in \mathbb{R}^n \mid A\mathbf{x} = \mathbf{0}\}.$$

To check whether V is a subspace of $\mathbb{R}^n$, we must verify the three conditions for being a subspace:

1. **Contains the zero vector**: The zero vector $\mathbf{0}$ satisfies $A\mathbf{0} = \mathbf{0}$, so $\mathbf{0} \in V$.

2. **Closed under addition**: If $\mathbf{x}, \mathbf{y} \in V$, then $A\mathbf{x} = \mathbf{0}$ and $A\mathbf{y} = \mathbf{0}$. Consider their sum:

$$A(\mathbf{x} + \mathbf{y}) = A\mathbf{x} + A\mathbf{y} = \mathbf{0} + \mathbf{0} = \mathbf{0}.$$

Thus, $\mathbf{x} + \mathbf{y} \in V$, so V is closed under addition.

3. **Closed under scalar multiplication**: For any scalar $c \in \mathbb{R}$ and any $\mathbf{x} \in V$, we have:

$$A(c\mathbf{x}) = cA\mathbf{x} = c\mathbf{0} = \mathbf{0}.$$

Hence, $c\mathbf{x} \in V$, so V is closed under scalar multiplication.

Since all three conditions hold, V is a subspace of $\mathbb{R}^n$.
N.B.: This subspace is called the null space or kernel of A, denoted as: $\ker(A) = \{\mathbf{x} \in \mathbb{R}^n \mid A\mathbf{x} = \mathbf{0}\}$

7. Consider three vectors,

$$u = \begin{bmatrix} 1 \\ 4 \\ 7 \end{bmatrix}, v = \begin{bmatrix} 2 \\ 5 \\ 8 \end{bmatrix}, w = \begin{bmatrix} 3 \\ 6 \\ 9 \end{bmatrix}$$

Is the vector $w \in \text{span}\{u, v\}$.

Answer: Yes. Check the system, $\alpha u + \beta v = w$.

$$\alpha \begin{bmatrix} 1 \\ 4 \\ 7 \end{bmatrix} + \beta \begin{bmatrix} 2 \\ 5 \\ 8 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \\ 9 \end{bmatrix}$$

$$\begin{bmatrix} \alpha + 2\beta \\ 4\alpha + 5\beta \\ 7\alpha + 8\beta \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \\ 9 \end{bmatrix}$$

Now, solve the system using the augmented matrix:

$$\left(\begin{array}{cc|c} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{array}\right)$$

8. (**Very important**) Consider three vectors,

$$u = \begin{bmatrix} 1 \\ 4 \\ 7 \end{bmatrix}, v = \begin{bmatrix} 2 \\ 5 \\ \lambda \end{bmatrix}, w = \begin{bmatrix} 3 \\ 6 \\ 9 \end{bmatrix}$$

For what value of λ the vector $w \in \text{span}\{u, v\}$.
Answer: We want w as linear combination of u, v,

$$\alpha u + \beta v = w$$

$$\alpha \begin{bmatrix} 1 \\ 4 \\ 7 \end{bmatrix} + \beta \begin{bmatrix} 2 \\ 5 \\ \lambda \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \\ 9 \end{bmatrix}$$

$$\begin{bmatrix} \alpha + 2\beta \\ 4\alpha + 5\beta \\ 7\alpha + \lambda\beta \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \\ 9 \end{bmatrix}$$

Get the augmented matrix:

$$\left(\begin{array}{cc|c} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & \lambda & 9 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & \lambda - 14 & -12 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & \lambda - 8 & 0 \end{array}\right)$$

Okay, the system is consistent if and only if $\lambda = 8$. Hence, when $\lambda = 8$, $w \in \text{span}\{u, v\}$ or can be written as the linear combination of u, v.

9. Find the basis and dimension of W, where

$$W = \text{span}\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 4 \\ 5 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \\ 7 \\ 9 \end{bmatrix} \right\}$$

10. Consider the previous question. Now, show that W can't span $\mathbb{R}^3$.

11. Show that the below vectors u, v, w in $\mathbb{R}^3$ are linearly independent.

$$u = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, v = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, w = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Answer: Check does the system $\alpha u + \beta v + \gamma w = 0$ give solution $\alpha = \beta = \gamma = 0$ or not:

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ -1 & 2 & 1 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 2 & 0 \end{array}\right)$$

Here, we can see that the solution is, $\alpha = \beta = \gamma = 0$. Hence, linearly independent.
Remark: Another approach is to show the determinant is not zero. (This method is only applicable because our matrix is a square one)

12. (**Very important**) Consider the following matrix,

$$A = \begin{bmatrix} 1 & -2 & 0 & -1 \\ -1 & 2 & 0 & 1 \\ 1 & -2 & 1 & 0 \end{bmatrix}$$

1. Calculate the basis of the Null(A) or NullSpace(A)
2. Calculate the basis of the Row(A) or RowSpace(A)
3. Calculate the basis of the Col(A) or ColSpace(A)
4. Find the rank
5. Find the nullity

Answer: (1) For NullSpace(A) find the general solution of $Ax = 0$ system:

$$\left(\begin{array}{cccc|c} 1 & -2 & 0 & -1 & 0 \\ -1 & 2 & 0 & 1 & 0 \\ 1 & -2 & 1 & 0 & 0 \end{array}\right)$$

N.B.: Just append $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ and apply Gaussian elimination.

Now,

$$\begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} =?$$

For rank, just count the pivots:

$$\begin{pmatrix} 1 & -2 & 0 & -1 \\ -1 & 2 & 0 & 1 \\ 1 & -2 & 1 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & -2 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Two pivots hence rank 2.

13. Consider the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^3$, where,

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x \\ y \\ x+y \end{bmatrix}$$

Find the standard matrix representation and then find $\operatorname{Ker}(T)$ and $\operatorname{Im}(T)$. And show that $\dim \operatorname{Ker}(T) + \dim \operatorname{Im}(T) = \dim \mathbb{R}^2$.
Answer: To find the standard matrix,

$$T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$
$$T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

Hence, the standard matrix is,

$$[A]_T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}$$

N.B.: To get the standard matrix, we always put the vectors in the **vertical form**.
To find the kernel just equate the Transformation output to zero vector and find the general solution:

$$\begin{bmatrix} x \\ y \\ x+y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

This system is very easy to solve because $x = 0, y = 0$. Hence, the general solution is:

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Wait, that's mean the $\operatorname{Ker}(T) = \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}$ which contain only one vector, hence $\dim \operatorname{Ker}(T) = 0$.

N.B.: There was no free variable, number of dimension was equal to the number of free variables. To find the rank apply gaussian elimination to the standard matrix:

$$\begin{bmatrix}1 & 0\\0 & 1\\1 & 1\end{bmatrix} \sim \begin{bmatrix}1 & 0\\0 & 1\\0 & 1\end{bmatrix} \sim \begin{bmatrix}1 & 0\\0 & 1\\0 & 0\end{bmatrix}$$

$\text{Rank}(T) = 2$. Hence, $\dim \text{Ker}(T) + \text{Rank}(T) = 0 + 2 = 2$.

14. Consider the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$, where,

$$T\left(\begin{bmatrix}x\\y\\z\end{bmatrix}\right) = \begin{bmatrix}x-y\\y-x\\x-z\end{bmatrix}$$

Find the standard matrix representation and then find $\text{Ker}(T)$ and $\text{Im}(T)$. And show that $\dim \text{Ker}(T) + \dim \text{Im}(T) = \dim \mathbb{R}^3$.

15. Consider the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$, where,

$$T\left(\begin{bmatrix}x\\y\\z\end{bmatrix}\right) = \begin{bmatrix}x\\y\\0\end{bmatrix}$$

Find the standard matrix representation and then find $\text{Ker}(T)$ and $\text{Im}(T)$. And show that $\dim \text{Ker}(T) + \dim \text{Im}(T) = \dim \mathbb{R}^3$.

16. Consider the linear transformation $T : \mathbb{R}^4 \to \mathbb{R}^3$, where,

$$T\left(\begin{bmatrix}x\\y\\z\\t\end{bmatrix}\right) = \begin{bmatrix}x-y+z+t\\2x-2y+3z+4t\\3x-3y+4z+5t\end{bmatrix}$$

Find the standard matrix representation and then find $\text{Ker}(T)$ and $\text{Im}(T)$. And show that $\dim \text{Ker}(T) + \dim \text{Im}(T) = \dim \mathbb{R}^4$.

Answer:

$$\text{Ker}(T) = \left\{\begin{bmatrix}1\\0\\-2\\1\end{bmatrix}, \begin{bmatrix}1\\1\\0\\0\end{bmatrix}\right\}, \quad \text{Im}(T) = \left\{\begin{bmatrix}1\\2\\3\end{bmatrix}, \begin{bmatrix}0\\1\\1\end{bmatrix}\right\}$$

Best of Luck!