# Linear Algebra & Fourier Analysis
# Quiz-01 (Solution)

1. (a) Solve the linear system (using Gaussian Elimination or Gauss-Jordan method):

$$\begin{cases} x + 3y + 2z &= 2 \\ 2x + 7y + 7z &= -1 \\ 2x + 5y + 2z &= 7 \end{cases}$$

**Hint:** You can reach row echelon form within 3 steps or reduced row echelon form within 6 steps.

**Solution:** Subtract twice the first equation from the second one and replace the second equation with the result to get

$$\left(\begin{array}{ccc|c} 1 & 3 & 2 & 2 \\ 0 & 1 & 3 & -5 \\ 2 & 5 & 2 & 7 \end{array}\right)$$

Subtract twice the first equation from the third one and replace the third equation with the result to get

$$\left(\begin{array}{ccc|c} 1 & 3 & 2 & 2 \\ 0 & 1 & 3 & -5 \\ 0 & -1 & -2 & 3 \end{array}\right)$$

Adding the last two equations

$$\left(\begin{array}{ccc|c} 1 & 3 & 2 & 2 \\ 0 & 1 & 3 & -5 \\ 0 & 0 & 1 & -2 \end{array}\right)$$

So, the system becomes,

$$\begin{cases} x + 3y + 2z = 2 \\ \qquad y + 3z = -5 \\ \qquad\qquad z = -2 \end{cases}$$

Now, using the back substitution we get:

$$x = 3, y = 1, z = -2.$$

(b) Suppose you have the following augmented matrix and the reduced row echelon form of that matrix respectively. Can you determine under what condition for $a, b$ and $c$ the system is consistent and inconsistent?

$$\left(\begin{array}{cccc|c} 1 & -1 & 1 & -3 & a \\ 1 & 1 & 3 & 1 & b \\ 1 & 0 & 2 & -1 & c \end{array}\right) \sim \cdots \sim \left(\begin{array}{cccc|c} 1 & 0 & 2 & -1 & c \\ 0 & 1 & 1 & 2 & -a+c \\ 0 & 0 & 0 & 0 & a+b-2c \end{array}\right)$$

**Solution:** Consistent if $a + b - 2c = 0$ and inconsistent if $a + b - 2c \neq 0$.

(c) Now, if we consider the columns as the vectors $v_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, $v_2 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$, $v_3 = \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}$ and $v_4 = \begin{pmatrix} -3 \\ 1 \\ -1 \end{pmatrix}$ in $\mathbb{R}^3$. Can $w = \begin{pmatrix} 13 \\ 3 \\ 8 \end{pmatrix}$ be written as a linear combination of $v_1, v_2, v_3$ and $v_4$?

**Hint:** Linear combination mean $w = \alpha v_1 + \beta v_2 + \gamma v_3 + \mu v_4$. Insert those vectors in this equation. Did you see something similar to $(b)$? What can you say if we consider $\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 13 \\ 3 \\ 8 \end{pmatrix}$.

**Solution**: The above equation becomes:

$$\alpha \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \beta \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + \gamma \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix} + \mu \begin{pmatrix} -2 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 13 \\ 3 \\ 8 \end{pmatrix}$$

$$\begin{pmatrix} \alpha \\ \alpha \\ \alpha \end{pmatrix} + \begin{pmatrix} -\beta \\ \beta \\ 0 \end{pmatrix} + \begin{pmatrix} \gamma \\ 3\gamma \\ 2\gamma \end{pmatrix} + \begin{pmatrix} -2\mu \\ \mu \\ -\mu \end{pmatrix} = \begin{pmatrix} 13 \\ 3 \\ 8 \end{pmatrix}$$

$$\begin{pmatrix} \alpha - \beta + \gamma - 2\mu \\ \alpha + \beta + 3\gamma + \mu \\ \alpha + 2\gamma - \mu \end{pmatrix} = \begin{pmatrix} 13 \\ 3 \\ 8 \end{pmatrix}$$

$$\underbrace{\begin{pmatrix} 1 & -1 & 1 & -2 \\ 1 & 1 & 3 & 1 \\ 1 & 0 & 2 & -1 \end{pmatrix}}_{Q1(b)} \begin{pmatrix} \alpha \\ \beta \\ \gamma \\ \mu \end{pmatrix} = \begin{pmatrix} 13 \\ 3 \\ 8 \end{pmatrix}$$

We already know when this is consistent by inspecting the right-hand side. Just check our $w$ satisfy the condition, $a + b - 2c = 13 + 3 - 16 = 0$. Hence, we can write $w$ as a linear combination.

(10+2+3 Marks)

2. Using Row Operations to find $A^{-1}$:

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{pmatrix}$$

(15 Marks)

**Solution:**

$$A^{-1} = \begin{pmatrix} -40 & 16 & 9 \\ 13 & -5 & -3 \\ 5 & -2 & -1 \end{pmatrix}$$

3. Write **True** if the statement is correct otherwise **False**. A short explanation is needed to justify your answer.

   (a) If a linear system has more unknowns than equations, then it has infinitely many solutions.

   **True**. Because in this case, we always have free variables.

   (b) If the number of equations in a linear system exceeds the number of unknowns, then the system must be inconsistent.

   **False**. Counterexample:

$$\begin{cases} x + y & = 2 \\ x - y & = 1 \\ 2x - 2y & = 2 \end{cases}$$

   (c) The linear system,

$$\begin{cases} x - y & = 3 \\ 2x - 2y & = k \end{cases}$$

   can't have a unique solution, regardless of the value of $k$.

   **True**, because either the lines will be the same or parallel as their slopes are the same. In both cases, we have either infinitely many solutions or no solution.

   (d) The linear system with corresponding augmented matrix

$$\left(\begin{array}{cc|c} 2 & -1 & 4 \\ 0 & 0 & -1 \end{array}\right)$$

   is consistent.

   **False**. The last equation give $0x + 0y = -1$ which is invalid.

(e) If $A$ is an $n \times n$ matrix that is not invertible, then the linear system $A\mathbf{x} = 0$ has infinitely many solution.

**True**. Not invertible of $A$ implies we have at least one row with zeros in our row echelon form. Which give us free variables.

(10 Marks)

## Best of Luck!