Final practice problem

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1. Find the general solution of the following system of linear questions,

$$\begin{cases} x + 3y + z + t &= 2 \\ 2x + 6y + 3z + 4t &= 5 \\ 7x + 21y + 8z + 9t &= 15 \end{cases}$$

   Answer:
$$\begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} = \begin{pmatrix} 1 - 3y - t \\ y \\ 1 - 2t \\ t \end{pmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + y \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 0 \\ -2 \\ 1 \end{bmatrix}$$

2. Calculate the inverse of the following matrix using Gauss-Jordan elimination (i.e. using reduced row echelon form),

$$A = \begin{pmatrix} 1 & 0 & -1 \\ 2 & 1 & -2 \\ 0 & 2 & 1 \end{pmatrix}$$

3. Is the set of vectors $W = \left\{ \begin{bmatrix} a \\ b \\ c \end{bmatrix} \in \mathbb{R}^3 \,\middle|\, a + b + c = 0 \right\}$ a subspace of the vector space $V = \mathbb{R}^3$.

   Answer: Yes. Check $u + v \in W$ and $\alpha u \in W$.

4. Let V be the set of all ordered pairs of real numbers and consider the following addition and scalar multiplication operations defined on the ordered pairs, $x = (x_1, x_2)$ and $y = (y_1, y_2)$:

$$x + y = (x_1 + x_2, y_1 + y_2 + 1), \quad \alpha \cdot x = (\alpha x_1, \alpha x_2)$$

   Answer: No. because $\alpha(u + v) \neq \alpha u + \alpha v$ where $u = (u_1, u_2), v = (v_1, v_2)$ and $\alpha \in \mathbb{R}$.

5. Consider the vector space V of 2 by 2 matrix, $M_{2\times 2}$. Now, Consider the following subsets:

   - $W_1 = \{M_{2\times 2} : \det M_{2\times 2} = 0\}$
   - $W_2 = \{M_{2\times 2} : \det M_{2\times 2} \neq 0\}$
   - $W_3 = \{M_{2\times 2} : \det M_{2\times 2} = 1\}$

Verify which of the subsets are subspace of V. Explain your answer.

Answer: None of them are subspace. Try to construct a counterexample from the following axioms: $\bar{0} + M = M$ and $1 \cdot M = M$.

6. Consider three vectors,

$$u = \begin{bmatrix} 1 \\ 4 \\ 7 \end{bmatrix}, v = \begin{bmatrix} 2 \\ 5 \\ 8 \end{bmatrix}, w = \begin{bmatrix} 3 \\ 6 \\ 9 \end{bmatrix}$$

Is the vector $w \in \text{span}\{u, v\}$.

Answer: Yes. Check the system, $\alpha u + \beta v = w$.

7. Find the row echelon form of the following augmented matrix:

$$\left(\begin{array}{cccc|c} 1 & -1 & 1 & -3 & a \\ 1 & 1 & 3 & 1 & b \\ 1 & 0 & 2 & -1 & c \end{array}\right)$$

Now, consider the vectors

$$v_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, v_2 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, v_3 = \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}, v_4 = \begin{pmatrix} -3 \\ 1 \\ -1 \end{pmatrix}, w = \begin{pmatrix} 13 \\ 3 \\ 8 \end{pmatrix}$$

Using the previous result can you determine whether w be written as a linear combination of v_1, v_2, v_3, v_4?

Answer: From row echelon we get:

$$\left(\begin{array}{cccc|c} 1 & -1 & 1 & -3 & a \\ 0 & 2 & 2 & 4 & b - a \\ 0 & 0 & 0 & 0 & \frac{-a-b}{2} + c \end{array}\right)$$

So, w can be written in terms of v_1, v_2, v_3, v_4 if $w = \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 13 \\ 3 \\ 8 \end{pmatrix}$ satisfy

$$\frac{-a-b}{2} + c = 0.$$

Remark: We can't use determinant here because the number of the vectors given to examine is not equal to the dimension of the vector space.

8. Find the basis and dimension of W, where

$$W = \text{span}\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 4 \\ 5 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \\ 7 \\ 9 \end{bmatrix} \right\}$$

9. Consider the previous question. Now, show that W can't span $\mathbb{R}^3$.

10. Show that whether $S = \{u, v, w, t\}$ span $\mathbb{R}^4$ or not?

$$u = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, v = \begin{bmatrix} 0 \\ 2 \\ 0 \\ 2 \end{bmatrix}, w = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}, t = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 2 \end{bmatrix}$$

Answer: To check take any arbitrary vector $\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$ from $\mathbb{R}^4$. Now, check does the system $\alpha u + \beta v + \gamma w + \mu t = \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$ is solvable or not:

$$\left(\begin{array}{cccc|c} 1 & 0 & 1 & 0 & a \\ 0 & 2 & 0 & 0 & b \\ 1 & 0 & 1 & 1 & c \\ 0 & 2 & 1 & 1 & d \end{array}\right) \sim \left(\begin{array}{cccc|c} 1 & 0 & 1 & 0 & a \\ 0 & 2 & 0 & 0 & b \\ 0 & 0 & 1 & 1 & d-b \\ 0 & 0 & 0 & 1 & c-a \end{array}\right)$$

As we have pivots in each row hence the system is solvable which implies u, v, w, t span $\mathbb{R}^4$.

Remark: Another approach is to show the determinant is not zero. (This method is only applicable because our matrix is a square one)

11. Show that the below vectors u, v, w in $\mathbb{R}^3$ are linearly independent.

$$u = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, v = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, w = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Answer: Check does the system $\alpha u + \beta v + \gamma w = 0$ give solution $\alpha = \beta = \gamma = 0$ or not:

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ -1 & 2 & 1 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 2 & 0 \end{array}\right)$$

Here, we can see that the solution is, $\alpha = \beta = \gamma = 0$. Hence, linearly independent.
Remark: Another approach is to show the determinant is not zero. (This method is only applicable because our matrix is a square one)

12. Consider the following matrix,

$$A = \begin{bmatrix} 1 & -2 & 0 & -1 \\ -1 & 2 & 0 & 1 \\ 1 & -2 & 1 & 0 \end{bmatrix}$$

   1. Calculate the basis of the Null(A) or NullSpace(A)
   2. Find the rank
   3. Calculate the basis of the Row(A) or RowSpace(A)
   4. Calculate the basis of the Col(A) or ColumnSpace(A)

Answer: (1) For NullSpace(A) find the general solution of $Ax = 0$ system:

$$\left(\begin{array}{cccc|c} 1 & -2 & 0 & -1 & 0 \\ -1 & 2 & 0 & 1 & 0 \\ 1 & -2 & 1 & 0 & 0 \end{array}\right)$$

Now,

$$\begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} =?$$

(2) For rank just count the pivots:

$$\begin{pmatrix} 1 & -2 & 0 & -1 \\ -1 & 2 & 0 & 1 \\ 1 & -2 & 1 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & -2 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Two pivots hence rank 2.

13. Consider the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^3$, where,

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x \\ y \\ x + y + z \end{bmatrix}$$

Find the standard matrix representation and then find Ker(T) and Im(T). And show that $\dim \text{Ker}(T) + \dim \text{Im}(T) = \dim \mathbb{R}^2$.

14. Consider the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$, where,

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x - y \\ y - x \\ x - z \end{bmatrix}$$

Find the standard matrix representation and then find $\mathrm{Ker}(T)$ and $\mathrm{Im}(T)$. And show that $\dim \mathrm{Ker}(T) + \dim \mathrm{Im}(T) = \dim \mathbb{R}^3$.

15. Consider the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$, where,

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x \\ y \\ 0 \end{bmatrix}$$

Find the standard matrix representation and then find $\mathrm{Ker}(T)$ and $\mathrm{Im}(T)$. And show that $\dim \mathrm{Ker}(T) + \dim \mathrm{Im}(T) = \dim \mathbb{R}^3$.

16. Consider the linear transformation $T : \mathbb{R}^4 \to \mathbb{R}^3$, where,

$$T\left(\begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix}\right) = \begin{bmatrix} x - y + z + t \\ 2x - 2y + 3z + 4t \\ 3x - 3y + 4z + 5t \end{bmatrix}$$

Find the standard matrix representation and then find $\mathrm{Ker}(T)$ and $\mathrm{Im}(T)$. And show that $\dim \mathrm{Ker}(T) + \dim \mathrm{Im}(T) = \dim \mathbb{R}^4$.

Answer:

$$\mathrm{Ker}(T) = \left\{ \begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right\}, \quad \mathrm{Im}(T) = \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$$

Best of Luck!